BRAID MONODROMY FACTORIZATION AND DIFFEOMORPHISM TYPES

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Abstract. In this paper we prove that if two cuspidal plane curves B_1 and B_2 have equivalent braid monodromy factorizations, then B_1 and B_2 are smoothly isotopic in $\mathbb{CP}^2$. As a consequence, we obtain that if the discriminant curves (or branch curves in other terminology) B_1 and B_2 of generic projections to $\mathbb{CP}^2$ of surfaces of general type S_1 and S_2, imbedded in a projective space by means of a multiple canonical class have equivalent braid monodromy factorizations, then S_1 and S_2 are diffeomorphic (if we consider them as real 4-folds).

§0. Introduction.

Let S be a non-singular algebraic surface in a projective space $\mathbb{CP}^r$ of $\deg S = N$. It is well-known that for almost all projections $pr : \mathbb{CP}^r \to \mathbb{CP}^2$ the restrictions $f : S \to \mathbb{CP}^2$ of these projections to S satisfy the following four conditions:

(i) f is a finite morphism of $\deg f = \deg S$;

(ii) f is branched along an irreducible curve $B \subset \mathbb{CP}^2$ with ordinary cusps and nodes, as the only singularities;

(iii) $f^*(B) = 2R + C$, where R is irreducible and non-singular, and C is reduced;

(iv) $f_{|R} : R \to B$ coincides with the normalization of B.

We will call such f *a generic morphism* and its branch curve will be called *the discriminant curve* of f.

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Two generic morphisms (S_1, f_1), (S_2, f_2) with the same discriminant curve B are said to be equivalent if there exists an isomorphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$.

The following assertion is known as Chisini's Conjecture.

Chisini's Conjecture. *Let B be the discriminant curve of a generic morphism $f : S \to \mathbb{CP}^2$ of $\deg f \geq 5$. Then f is uniquely determined by the pair $(\mathbb{CP}^2, B)$.*

It is easy to see that the similar conjecture for generic morphisms of projective curves to $\mathbb{CP}^1$ is not true. On the other hand in [Kul1] it is shown that Chisini's Conjecture holds for the discriminant curves of almost all generic morphisms of any projective surface. In particular, if S is any surface of general type with ample canonical class, then Chisini's Conjecture holds for the discriminant curves of the generic morphisms $f : S \to \mathbb{CP}^2$ given by a three-dimensional linear subsystem of the m-canonical class of S, where $m \in \mathbb{N}$. The discriminant curve of such a generic morphism will be called *m-canonical discriminant curve.*

Let B be an algebraic curve in $\mathbb{CP}^2$ of degree p. Topology of the embedding $B \subset \mathbb{CP}^2$ is determined by the *braid monodromy* of B which is described by a factorization of the "full-twist" Δ_p^2 in the semi-group B_p^+ of the braid group B_p of p string braids (in standard generators, $\Delta_p^2 = (X_1 \cdot \ldots \cdot X_{p-1})^p$). If B is a cuspidal curve, then this factorization can be written as follows

$$\Delta_p^2 = \prod_i Q_i^{-1} X_1^{\rho_i} Q_i, \qquad \rho_i \in (1, 2, 3), \tag{1}$$

where X_1 is a positive half-twist in B_p.

Let

$$h = g_1 \cdot \ldots \cdot g_r \tag{2}$$

be a factorization in B_p^+. The transformation which changes two neighbouring factors in (2) as follows:

$$g_i \cdot g_{i+1} \longmapsto (g_i g_{i+1} g_i^{-1}) \cdot g_i,$$

or

$$g_i \cdot g_{i+1} \longmapsto g_{i+1}(g_{i+1}^{-1} g_i g_{i+1})$$

is called a *Hurwitz move.*

For $z \in B_p$, we denote

$$h_z = z^{-1} g_1 z \cdot z^{-1} g_2 z \cdot \ldots \cdot z^{-1} g_r z$$

and say that the factorization expression h_z is obtained from (2) by simultaneous conjugation by z. Two factorizations are called *Hurwitz and conjugation equivalent* if one can be obtained from the other by a finite sequence of Hurwitz moves followed by a simultaneous conjugation. We will say that two factorizations of the form (1) belong to the same *braid factorization type* if they are Hurwitz and conjugation equivalent. The main problems in this direction are the following:

Problem 1. *Let $B \subset \mathbb{CP}^2$ be a cuspidal curve. Does the braid factorization type of the pair $(\mathbb{CP}^2, B)$ uniquely determine the diffeomorphic type of this pair, and vice versa?*

Problem 2. *Let $\Delta_p^2 = \mathcal{E}_1$ and $\Delta_p^2 = \mathcal{E}_2$ be two braid monodromy factorizations. Does there exist a finite algorithm to recognize whether these two braid monodromy factorizations belong to the same braid factorization type?*

One of the main results of this article is

Theorem 1. *Let B_1, $B_2 \subset \mathbb{CP}^2$ be two cuspidal algebraic curves. Assume that the pairs $(\mathbb{CP}^2, B_1)$ and $(\mathbb{CP}^2, B_2)$ have the same braid factorization type. Then the pairs $(\mathbb{CP}^2, B_1)$ and $(\mathbb{CP}^2, B_2)$ are diffeomorphic.*

It is well-known that there exist four-dimensional smooth manifolds which are homeomorphic, but not diffeomorphic. One of the most important problems in four-dimensional geometry is to find invariants which distinguish smooth structures on the same topological four-dimensional manifold. We believe that in the

algebraic case one can use the braid factorization type of the discriminant curve of a generic morphism of a projective surface S to $\mathbb{CP}^2$ as an invariant of the smooth structure (induced by complex structure) on S, considered as a four-dimensional real manifold. We will prove

Theorem 2. *Let $f_1 : S_1 \to \mathbb{CP}^2$ and $f_2 : S_2 \to \mathbb{CP}^2$ be two generic morphisms of non-singular projective surfaces, and let $B_1,\ B_2 \subset \mathbb{CP}^2$ be their discriminant curves. Assume that Chisini's Conjecture holds for $(\mathbb{CP}^2, B_1)$. If the pairs $(\mathbb{CP}^2, B_1)$ and $(\mathbb{CP}^2, B_2)$ have the same braid factorization type, then S_1 and S_2 are diffeomorphic.*

Corollary. *Let S_1 and S_2 be two surfaces of general type with ample canonical class and let B_1 and B_2 be m-canonical discriminant curves, respectively, of generic morphisms $f_1 : S_1 \to \mathbb{CP}^2$ and $f_2 : S_2 \to \mathbb{CP}^2$ given by three-dimensional linear subsystem of m-canonical class on S_i, where $m \in \mathbb{N}$. If the pairs $(\mathbb{CP}^2, B_1)$ and $(\mathbb{CP}^2, B_2)$ have the same braid factorization type, then S_1 and S_2 are diffeomorphic.*

Here is a brief summary of the rest of this article. In Sections 1 - 4, we recall definitions and some facts on the braid monodromy technique developed by B. Moishezon and the second author. Section 5 is devoted to the description of generators of the centralizer of the multiple half-twists in the braid group. This description is a key to the proof of Theorem 1. In Section 6, we recall or prove some assertions (which may be well-known) on smooth isotopy of manifolds which we will use in the proof of the main results. Section 7 is devoted to the proof of Theorem 1, and in Section 8 we prove Theorem 2.

§1. Braid monodromy of an affine curve.

Throughout this paper, we use the following notations:

S is a curve in $\mathbb{C}^2$, $p = \deg S$,

$\pi : \mathbb{C}^2 \to \mathbb{C}$ a generic projection on the first coordinate,

$K(x) = \{y \mid (x,y) \in S\}$ ($K(x) =$ projection to y-axis of $\pi^{-1}(x) \cap S$),

$N = \{x \mid \#K(x) \lneqq p\}$,

$M' = \{x \in S \mid \pi \mid_X$ is not étale at $x\}$ $(\pi(M') = N)$.

Assume $\#(\pi^{-1}(x) \cap M^1) = 1$ for all $x \in N$.

Let E (resp. D) be a closed disk on x-axis (resp. y-axis) s.t. $M' \subset E \times D$, $N \subset \mathrm{Int}(E)$, and such that $\pi_{|(E\times D)\cap B}$ is a proper morphism of degree p.

We choose $u \in \partial E$ and put $K = K(u) = \{q_1, \ldots, q_p\}$.

In such a situation, we will introduce the concept of "braid monodromy."

Definition. Braid monodromy of B with respect to $E \times D$, π, u.

Every loop $\gamma : [0,1] \to E \setminus N$ starting at u has liftings to a system of p paths in $(E \setminus N) \times D$ starting at $q_1, \ldots, q_p$. Projecting them to D we get p paths in D defining a motion $\{q_1(t), \ldots, q_p(t)\}$ of p points in D starting and ending at K.

This motion defines a braid in $B_p[D, K]$, (see [MoTe1], Section III). Thus we get a map $\varphi : \pi_1(E \setminus N, u) \to B_p[D, K]$. This map is evidently a group homomorphism, and it is the braid monodromy of B with respect to $E \times D$, π, u. We sometimes denote φ by φ_u.

Definition. Braid monodromy of B with respect to π, u.

When considering the braid induced from the previous motion as an element of the group $B_p[\mathbb{C}_u, K]$ we get the homomorphism $\varphi : \pi_1(E \setminus N, u) \to B_p[\mathbb{C}_u, K]$ which is called the braid mondromy of B with respect to π, u. We sometimes denote φ by φ_u.

In order to present an example of a braid monodromy calculation, we recall a geometric model of the braid group and the definition of a half-twist.

Definition. Braid group $B_p[D, K]$

Let D be a closed disc in $\mathbb{R}^2$, $K \subset D$, K finite. Let $\mathcal{B}$ be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}_{\partial D}$. For $\beta_1, \beta_2 \in \mathcal{B}$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of

$\pi_1(D \setminus K, u)$. The quotient of B by this equivalence relation is called the braid group $B_p[D, K]$ ($p = \#K$). We sometimes denote by $\overline{\beta}$ the braid represented by β. The elements of $B_p[D, K]$ are called braids.

Definition. <u>$H(\sigma)$, half-twist defined by σ</u>

Let D, K be as above. Let $a, b \in K$, $K_{a,b} = K \setminus \{a, b\}$ and σ be a simple (i.e. without self-intersections) path in $D \setminus \partial D$ connecting a with b such that $\sigma \cap K = \{a, b\}$. Choose a small regular neighbourhood U of σ, $K_{a,b} \cap U = \emptyset$, and an orientation preserving diffeomorphism $\psi : \mathbb{R}^2 \longrightarrow \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with usual "complex" orientation) such that $\psi(\sigma) = [-1, 1] = \{z \in \mathbb{C}^1 \mid \mathrm{Re}\, z \in [-1, 1], \mathrm{Im} z = 0\}$ and $\psi(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$. Let $\alpha(r), r \geqslant 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geqslant 2$. Define a diffeomorphism $h : \mathbb{C}^1 \to \mathbb{C}^1$ as follows: for $z \in \mathbb{C}^1$, $z = re^{i\varphi}$ let $h(z) = re^{i(\varphi + \alpha(r)\pi)}$. It is clear that the restriction of h to $\{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\}$ coincides with the positive rotation on π, and that the restriction to $\{z \in \mathbb{C}^1 \mid |z| \geq 2\}$ is the identity map. The diffeomorphism $H(\sigma) = \psi^{-1} \circ h \circ \psi$ will be called a half-twist.

A half-twist $H(\sigma)$ defines *a geometric braid* $\overline{\sigma}$ (i.e. p paths without self-intersections in $D \times [0, 1]$ starting at $K \times \{0\}$ and ending at $K \times \{1\}$, $K = \{q_1, \dots, q_p\}$). This braid can be presented as

$$\begin{aligned} &(\delta_j(t), t) = (q_i, t) && \text{if } q_j \neq a, b;\\ &(\delta_j(t), t) = (\psi^{-1}(e^{\pi i t}), t) && \text{if } q_j = a;\\ &(\delta_j(t), t) = (\psi^{-1}(-e^{\pi i t}), t) && \text{if } q_j = b. \end{aligned}$$

The following is the basic braid monodromy associated to a single singularity of an algebraic curve.

Proposition - Example 1.1. *Let* $E = \{x \in \mathbb{C} \mid |x| \leq 1\}$, $D = \{y \in \mathbb{C} \mid y \leq R\}$, $R \gg 1, B$ *is the curve* $y^2 = x^\nu$, $u = 1$. *Clearly, here* $n = 2$, $N = \{0\}$, $K =$

$\{-1,+1\}$ and $\pi_1(E-N,1)$ is generated by $\Gamma = \partial E$ (positive orientation). Denote by $\varphi : \pi_1(E \setminus N, 1) \to B_2[D,K]$ the braid monodromy of B with respect to $E \times D$, π, u.

Then $\varphi(\Gamma) = H^\nu$, where H is the positive half-twist defined by $[-1,1]$ ("positive generator" of $B_2[D,K]$).

Proof. We can write $\Gamma = \{e^{2\pi it}, t \in [0,1]\}$. Lifting Γ to S we get two paths:

$$\begin{aligned} \delta_1(t) &= \left(e^{2\pi it},\ e^{2\pi i\nu t/2}\right) \\ \delta_2(t) &= \left(e^{2\pi it},\ -e^{2\pi i\nu t/2}\right). \end{aligned}$$

Projecting $\delta_1(t)$, $\delta_2(t)$ to D we get two paths:

$$\begin{aligned} a_1(t) &= e^{\pi it\cdot\nu}, & 0 \le t \le 1 \\ a_2(t) &= -e^{\pi it\cdot\nu}, & 0 \le t \le 1. \end{aligned}$$

These paths induce a motion of {1, -1} in D. This motion is the ν-th power of the motion $\mathcal{M}$:

$$\begin{aligned} b_1(t) &= e^{\pi it}, & 0 \le t \le 1 \\ b_2(t) &= -e^{\pi it}, & 0 \le t \le 1. \end{aligned}$$

The braid of $B_2[D, \{1,-1\}]$ induced by $\mathcal{M}$ coincides with the half-twist H corresponding to $[-1,1] \subset D$. Thus $\varphi(\Gamma) = H^\nu$. □

We recall the notion of a geometric free base of the fundamental group of a punctured disc and a basic property of the free base.

Definition. A bush.

Let E, $N = \{u_1, \dots, u_n\}$, u be as above. Consider in E ordered sets of simple paths $(T_1, \dots, T_n)$ connecting u_i's with u such that

(1) $T_i \cap T_j = u$ if $i \neq j$;

(2) for a small circle $c(u)$ around u each $T_i \cap c(u)$ is a single point, namely w_i, and the order in $(w_1, \dots, w_n)$ is consistent with the positive ("counterclockwise") orientation of $c(u)$.

Let c_j be the boundary of a closed disc E_j of small radius with center at u_j. Denote by $T'_j = T_j \setminus (T_j \cap E_j)$ and $l(T_j) = T'_j \cdot c_j \cdot {T'_j}^{-1}$ a loop (and the corresponding element in $\pi_1(E \setminus N, u)$) in which a point moves counterclockwise along c_j.

We say that two such sets $(T_1, \dots, T_n)$ and $(\widetilde{T}_1, \dots, \widetilde{T}_n)$ are equivalent if

$$\ell(T_i) = \ell(\widetilde{T}_i) \qquad (\text{in } \pi_1(E \setminus N, u)$$

for all $i = 1, \dots, n$. An equivalence class of such sets is called a bush in $(E \setminus N, u)$. The bush represented by $(T_1, \dots, T_n)$ is denoted by $\langle T_1, \dots, T_n \rangle$.

Definition. <u>geometric base, g − base</u>

Let E, N, u be as above. A g–base of $\pi_1(E \setminus N, u)$ is an ordered free base of $\pi_1(E \setminus N, u)$ which has the form $(\ell(T_1), \dots, \ell(T_n))$ where $\langle T_1, \dots, T_n \rangle$ is a bush in $E \setminus N$.

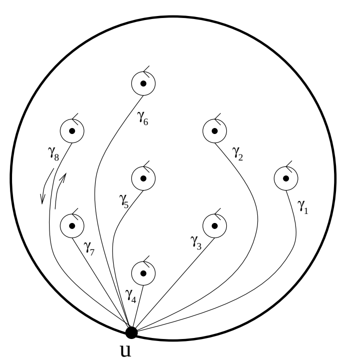

Fig. 1.1

Proposition-Definition. *Denote by $\wp$ the element of $\pi_1(E \setminus N, u)$ represented by the loop ∂E (with positive orientation). There exists a unique element of $B_n[E, N]$, denoted by Δ_n^2 or $\Delta_n^2[E, N]$ such that for any g-base $\Gamma_1, \dots, \Gamma_n$ of $\pi_1(E \setminus N, u)$*

$$(\Gamma_i)\Delta_n^2 = \wp \Gamma_i \wp^{-1}.$$

Proof. [MoTe1], V.2.1.

Remark. Clearly, Δ_p^2 acts as a full-twist around all the points of N.

Proposition 1.2. $\Delta_n^2 \in \text{Center}\, B_n[E,N]$.

Proof. [MoTe1], V.4.1.

Proposition - Example 1.3. *Let B be a union of p lines, meeting in one point s_0, $s_0 = (x(s_0), y(s_0))$. Let D, E, u, $K = K(u)$ be as in the beginning of §1. Let φ be the braid monodromy of B with respect to $E \times D$, π, u. Clearly, here $N =$ single point $x(s_0)$ and $\pi_1(E \setminus N, u)$ is generated by $\Gamma = \partial E$. Then $\varphi(\Gamma) = \Delta_p^2 = \Delta_p^2\,[D, K(u)]$.*

Proof. By a continuous change of s_0 and the n lines passing through s_0 (and by uniqueness of Δ_p^2) we can reduce the proof to the following case: $B = \cup L_k$, $L_k\colon\ y = j_k x,\ j_k = e^{2\pi i k/p}, \quad k = 0, \dots, p-1$. Then $N = \{0\}$. We can take $E = \{c||x| \le 1\}$, $u = 1$, $\Gamma = \partial E = \{x = e^{2\pi i t},\, t \in [0,1]\}$. Lifting ∂E to B and then projecting it to D, we get p loops:

$$a_k(t) = e^{2\pi i (t + k/p)}, \quad k = 0, \dots, p-1, \quad t \in [0,1].$$

Thus the motion of $a_k(0)$ represented by $a_k(t)$ is a full-twist which defines the braid $\Delta_p^2\,[D, \{a_k(0)\}] = \Delta_p^2\,[D, K(1)]$.

(To check this last fact, see the corresponding actions in $\pi_1\,(D \setminus K, u)$).

Definition. Frame of $B_p[D,K]$.

Let D, $K = \{q_1, \dots q_p\}$ be as in the beginning of §1. Let us choose a system of simple smooth paths $(\sigma_1, \dots, \sigma_{p-1})$ in $D \setminus \partial D$ such that σ_i connects q_i with q_{i+1} and $L = \cup \sigma_j$ is a simple smooth path. The ordered system of half-twists $(H_1, \dots, H_{p-1})$ defined by $\{\sigma_i\}_{i=1}^{p-1}$ is called a frame of $B_p[D,K]$. Sometimes such a system of paths $\{\sigma_i\}_{i=1}^{p-1}$ will also be called a frame of $B_p[D,K]$.

Theorem 1.4. *Let $(H_1, \dots, H_{p-1})$ be a frame of $B_p[D,K]$. Then $B_p[D,K]$ is generated by $H_1, \dots, H_{p-1}$.*

Proof. See, for example, [MoTe1].

Definition. <u>Generalized half-twist $\Delta_{i,j}$ given by a skeleton $L_{i,j} = \bigcup_{r=i}^{j-1} \sigma_r$.</u>

In the notation of the above definition let $\{\sigma_r\}_{r=1}^{p-1}$ be a frame of $B_p[D,K]$ and let $i < j$. Choose a small neighbourhood U of the path $L_{i,j}$ and an orientation preserving diffeomorphism $\psi : \mathbb{R}^2 \longrightarrow \mathbb{C}^1$ such that

(1) $\psi(L_{i,j}) = [-1,1] = \{z \in \mathbb{C}^1 \mid \mathrm{Re}\, z \in [-1,1], \mathrm{Im}\, z = 0\}$;

(2) $\psi(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$;

(3) the set $\{\psi(q_r)\}_{r=i}^{j}$ is invariant relative to the involution $\mathrm{Re}\, z \mapsto -\mathrm{Re}\, z$.

Let $\alpha(r), r \geqslant 0$ be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geqslant 2$. Define a diffeomorphism $h : \mathbb{C}^1 \to \mathbb{C}^1$ as follows: for $z \in \mathbb{C}^1$, $z = re^{i\varphi}$, let $h(z) = re^{i(\varphi + \alpha(r)\pi)}$. The diffeomorphism $\Delta_{i,j} = \psi^{-1} \circ h \circ \psi$ will be called a half-twist given by $L_{i,j} = \cup_{r=i}^{j-1} \sigma_r$.

Remark. The full-twist $\Delta_{i,j}^2 = (H_i \cdot \dots \cdot H_{j-1})^{j-i+1}$. In particular, $\Delta_{i,i+1} = H_i$ and $\Delta_{1,p}^2 = \Delta_p^2$, where the the half-twist H_i and the full-twist $\Delta_p^2 = \Delta_p^2[D,K]$ was defined above.

§2. Positivity of the braid monodromy.

In this section, we shall show that the braid monodromy takes, in fact, values in the semigroup B_p^+ of positive braid monodromy generated by the (counterclockwise) half-twists.

We keep the same notations as in §1, and introduce additional ones.

Definition. <u>ψ_T, Lefschetz diffeomorphism induced by a path T</u>

Let T be a path in $E \setminus N$ connecting x_0 with x_1, $T : [0,1] \to E \setminus N$. There exists a continuous family of diffeomorphisms $\psi_{(t)} : D \to D$, $t \in [0,1]$, such that $\psi_{(0)} = Id$, $\psi_{(t)}(K(x_0)) = K(T(t))$ for all $t \in [0,1]$, and $\psi_{(t)}(y) = y$ for all $y \in \partial D$.

For emphasis we write $\psi_{(t)} : (D, K(x_0)) \to (D, K(T(t))$. Lefschetz diffeomorphism induced by a path T is the diffeomorphism

$$\psi_T = \psi_{(1)} : (D, K(x_0)) \underset{\sim}{\to} (D, K(x_1)).$$

Since $\psi_{(t)}\,(K(x_0)) = K(T(t))$ for all $t \in [0,1]$, we have a family of canonical isomorphisms

$$\psi_{(t)}^{\nu} : B_p\,[D, K(x_0)] \underset{\sim}{\to} B_p\,[D, K(T(t))]\,, \qquad \text{for all } t \in [0,1].$$

Definition. $\underline{L_T,\ \text{Lefschetz isomorphism induced by } T}$

$$L_T = \psi_T^{\nu} = \psi_{(1)}^{\nu} : B_p\,[D, K(x_0)] \underset{\sim}{\to} B_p\,[D, K(x_1)]$$
$$(\overline{\beta})L_T = \overline{\psi_T^{-1} \circ \beta \circ \psi_T}.$$

It is easy to check that L_T depends only on the homotopy class of T.

Notation. $\underline{\psi_{T,B}\quad L_{T,B}}$

L_T and ψ_T depend not only on T, but also on B. To avoid confusion, we shall emphasize and use $\psi_{T,B}$ and $L_{T,B}$.

Remark. There is an equivalent definition of braid monodromy $\varphi_u : \pi_1(E \setminus N, u) \to B_p[D, K]$. Take any $\delta \in \pi_1(E \setminus N, u)$. Let $\underline{\delta}$ be a loop representing δ. Then $\varphi(\delta)$ is the braid represented by the diffeomorphism $\psi_{\underline{\delta}}$.

Proof. Both $\psi_{\underline{\delta}}$ and $\varphi(\delta)$ are induced from "pushing" along Γ.

Lemma 2.1.

(1) $\psi_{T_1 T_2} = \psi_{T_1} \circ \psi_{T_2}$.

(2) $H(\sigma)L_t = H((\sigma)\psi_T)$.

(3) *If T is a closed loop, then ψ_T defines a braid $\overline{\psi}_T$ in B_p and $(b)L_T = \overline{\psi}_T^{-1} b \overline{\psi}_T$ (composition from left to right).*

Proof. (1) and (2) are proved immediately.

(3) Assume $b = \overline{\beta}$. By definition of L_T, $(\overline{\beta})L_T = \overline{\psi_T^{-1} \circ \beta \circ \psi_T}$. Since ψ_T and β each defines a braid, $(b)L_T = \overline{\psi_T}^{-1} \circ \overline{\beta} \circ \overline{\psi}_T = \overline{\psi_T}^{-1} b \overline{\psi}_T$. □

Let $M' = \{s_j\}_{j=1}^n$ be the singular points of $\pi \big|_B$, $\pi(M') = N$. For every $j = 1, \dots, n$, let D'_j be a small disk on y-axis centered at $y(s_j)$ such that $D_j \subseteq D$ and that $(x(s_j) \times D'_j \cap B = s_j$. Let E'_j be a sufficiently small closed neighborhood of $x_j = x(s_j)$ on the x-axis such that $E'_j \cap N = x_j$ and the number $\#(x \times \mathrm{Int}(D'_j) \cap B)$ is independent of x for all $x \in E'_j \setminus \{x_j\}$. We call this number the local degree of π at s_j or $\deg_{s_j} \pi$. Let $m_j = \deg_{s_j} \pi$. Choose a point $x'_j \in \partial E$. Let $K'(x'_j) = K(x'_j) \cap D'_j$.

Definition. $\underline{\psi_{T,s_j}\text{, Lefschetz embedding induced by } s_j \text{ and } T}$

Let $T : [0,1] \to \mathbb{C}$ be a path in $E \setminus (N \cup (\mathrm{Int}\ E'_j))$ connecting x'_j to a point $u' \in E \setminus N$. The diffeomorphism $\psi_{T,s_j} = \psi_T \big|_{D'_j}$: $(D'_j, K'(x'_j)) \to (D, K(u'))$ is called Lefschetz embedding induced by s_j and T, where ψ_T is the Lefschetz diffeomorphism induced by T.

Remark. Let $m_j = \deg_{s_j} \pi$. Take m_j liftings of T to B starting at the different points of $K'(x'_j)$. These liftings are real curves in $T \times D$. We can think of ψ_T as "pullings" of $K'(x'_j)$ in $T \times D$ along these real curves.

Definition. $\underline{L_{T,s_j}\text{, Lefschetz injection induced by } T}$

Let $s_0 \in N$. Let D'_0, x_0, E_0, m_0, u' be as in the definition of Lefschetz embedding. Consider $\psi_{T,s_j} : (D'_j \times K(x'_j)) \to (D, K(u'))$ a Lefschetz embedding induced by s_j and T. We have $\psi_{T,s_j}(K'(x'_j)) \subset K(u')$ and $(K(u') \setminus \psi_{T,s_j}(K(x'_j)) \cap \psi_{T,s_j}(\mathrm{Int}\, D'_j) = \phi$. The Lefschetz injection induced by T is the canonical injection induced from ψ_{T,s_j}, $L_T = L_{T,s_j} = \psi_T^{\nu} : B_{m_j}\left[D'_j, K'(x'_j)\right] \hookrightarrow B_p[D, K(u')]$ which is is well-defined by the above inclusions.

To compute the braid mondromy, we need to know $\{\varphi(\delta_j)\}_{j=1}^n$ for the g-base $\{\delta_j = \ell(\gamma_j)\}_{j=1}^n$ defined by the bush $\{\gamma_j\}_{j=1}^n$ in $(E \setminus N, u)$. We actually need to be

able to compute $\varphi(\delta_j)$ for such $\{\delta_j\}$ since it is the basic data for any applications of braid monodromy. We can represent each δ_j as $\tilde{\gamma}_j^{-1} \circ \partial E'_j \circ \tilde{\gamma}_j$ (E'_j was defined earlier). Thus, to know $\varphi(\delta_j)$ it is sufficient to know the Lefschetz injection $L_{\tilde{\gamma}_j}$: $B_{m_j}[D'_j, K'(x'_j)] \to B_p[D, K]$ and also the local braid monodromy φ_{s_j} of $B \cap (E'_j \times D'_j)$ with respect to $E'_j \times D'_j$, π,x'_j as defined here.

Definition. $\underline{\varphi_{s_j} \text{ local braid monodromy of } B \text{ at } s_j}$

The local braid monodromy of B at s_j is

$$\varphi_{s_j} : \pi_1(E'_j \setminus \{x_j\}, x'_j) \to B_{m_j}\left[D'_j, K'(x'_j)\right],$$

the braid monodromy of $B \cap [E'_j \times D'_j]$ with respect to $E'_j \times D'_j$, π, x'_j ($K'(x'_j) = K(x'_j) \cap D'_j$).

It is clear that φ_{s_j} is determined only by $\varphi_{s_j}(\partial E'_j)$. The following lemma is evident.

Lemma 2.2. *Let T be any path in $E \setminus (N \cup (\text{Int } E'_j))$ connecting x'_j with u and $\delta = T^{-1} \circ \partial E'_j \circ T = \ell(T)$. Then $\varphi(\delta) = \left(\varphi_{s_j}(\partial E'_j)\right) L_T$. In particular, $\varphi(\delta_j) = \left(\varphi_{s_j}(\partial E'_j)\right) L_{\tilde{\gamma}_j}$.*

Remark. The Lemma actually indicates that the braid monodromy φ is completely determined if we know local braid monodromies and Lefschetz injections for some bush in $E \setminus N$.

Lemma 2.3. *Let s_j be a singularity of B which is locally presented by $y^2 = x^\nu$, that is, $m_j = \deg_{s_j} \pi = 2$. Then $\varphi(\delta_j) = (H_j)^\nu$, where H_j is a positive half-twist defined by some path σ, and in particular it is a positive braid.*

Proof. Follows from Proposition - Example 1.1. and Lemma 2.2.

Proposition 2.4. *Let $\varphi : \pi_1(E \setminus N, u) \to B_p[D, K]$ be braid monodromy of a curve B, $\{\delta_j\}$ a g-base of $\pi_1(E \setminus N, u)$. Then all $\varphi(\delta_j) \in B_p^+ = B_p^+[D, K]$.*

Proof. Given a curve B, we can find a curve $B^{(1)}$ close enough to B, nonsingular and of the same degree. Let $K^{(1)} = \{y \mid (u, y) \in B^{(1)}\}$ and $M^{(1)}$, $N^{(1)}$ be as in §1. We can naturally identify $B_p[D, K]$ with $B_p[D, K^{(1)}]$. Each $s_j \in M$ splits into a finite set of singular points $\{s_{j_i}\} \subseteq M^{(1)}$ which locally are of type $x = y^2$. Each $x_j \in N$ will split into points $\{x_{j_i}\} = \{\pi(s_{j_i})\} \subseteq N^{(1)}$. Clearly, $N^{(1)} = \{x_{ji}\}_{j,i}$. Let $\varphi^{(1)}$ be the braid monodromy of $B^{(1)}$ with respect to $E \times D$, π, u, $N^{(1)}$. We can find $\{\delta_{j_i}\}$ a g-base of $\pi_1(E \setminus N^{(1)}, u)$ such that each $\delta_j = \prod\limits_i \delta_{j_i}$. Natural identification of $B_p[D, K]$ and $B_p[D, K^{(1)}]$ will give us that each $\varphi(\delta_j) = \prod\limits_i \varphi^{(1)}(\delta_{j_i})$ (we use the fact that $B^{(1)}$ is very close to B). By Lemma 2.3, each $\varphi^{(1)}(\delta_{j_i})$ is a positive half-twist. Thus, each $\varphi(\delta_j) = \prod\limits_i \varphi^{(1)}(\delta_{j_i}) \in B_p^+$. □

§3. Braid monodromy of a projective curve.

Definition. <u>Braid monodromy of a projective curve</u>

Let B be an algebraic curve of degree p in $\mathbb{CP}^2$. Choose generically a line L at infinity $(\#(L \cap B) = p)$ and affine coordinates (x, y) in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L$, so that the projection $\pi(x, y) = x$ on x-axis of the curve $B \cap \mathbb{C}^2$ is generic (in particular, the center of this projection in $\mathbb{CP}^2$ must be outside of B). Let $\pi(x, y) = x$. Let $N = \{x \in \mathbb{C} \mid \pi^{-1}(x) \cap B \lneqq p\}$, E is a closed disk on the x-axis with $N \subset \mathrm{Int}(E)$, D is a closed disk on the y-axis with $\pi^{-1}(E) \cap B \subset E \times D$. Choose $u \in \partial E$.

The braid monodromy of B with respect to L, u is the braid monodromy of $B \cap (E \times D)$ with respect to $E \times D$, π, u, i.e., the homomorphism

$$\varphi : \pi_1(E \setminus N, u) \to B_p[D, K]$$

defined in §1.

Proposition 3.1. *Let B be an algebraic curve of degree p in $\mathbb{CP}^2$. Let L, π, u, D, E, $K(u)$ be as above. Let φ be the braid monodromy of B with respect to L, π,*

u. Let $\delta_1, \dots, \delta_q$ be a g-base of $\pi_1(E \setminus N, u)$. Then

$$\prod_{i=1}^{q} \varphi(\delta_i) = \Delta_p^2 = \Delta_p^2[u \times D, K(u)].$$

Proof. Because $\prod_{i=1}^{q} \delta_i = \partial E$ (positive oriented), we have to prove that $\varphi(\partial E) = \Delta_p^2$. We can assume E arbitrarily big, so that ∂E will be very close to ∞ at the x-axis. Continuously deforming coefficients of the equations of B such that new curves will be transversal to L_∞ we can reduce the proof to an equation which defines union of n lines intersecting at a single point. Now use Proposition - Example 1.3. □

Lemma 3.2. $\Delta_p^2 \in B_p^+$.

Proof. Proposition 2.4.

Definition. Braid monodromy factorization (associated to projective curve)

Braid monodromy factorization associated to a plane projective curve is a product of the form $\Delta_p^2 = \prod_i \varphi(\delta_i)$, where φ is the braid monodromy of the projective curve and $\{\delta_j\}$ is a g-base of $\pi_1(E \setminus N_1, u)$.

Remarks.

(1) A g-base of $\pi_1(E \setminus N, u)$ and the corresponding product-form determine the braid monodromy. For applications it is usually sufficient to know a product-form of a braid monodromy without reference to a particular g-base.

(2) The product-form is not a prime factorization of Δ_p^2 unless B is nonsingular. For a nonsingular B, each $\varphi(\delta_i)$ is a positive half-twist which is a prime element of B_p^+.

Proposition 3.3. *Let B be a (generalized) cuspidal curve on $\mathbb{CP}^2$ (that is, all singularities of B are locally given by $y^2 = x^\nu$, $\nu \in \mathbb{N}$). Then any product-form of the braid monodromy of δ can be written as $\Delta_p^2 = \prod_i (Q_i^{-1} H_1^{\nu_i} Q_i)$ where H_1 is a positive half-twist and each $\nu_i \in \mathbb{N}$.*

Proof. Recall that we are using generic projections of $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}$ with respect to the projective curves. Each singularity of $\pi|_B$ is of the type $y^2 = x^\nu$, $\nu \in \mathbb{N}$. Now use Lemma 2.3 to get $\varphi(\delta_i) = H_i^{\nu_i}$, where H_i is a half-twist. Every two half-twists in B_p are conjugate, so for all i there exist Q_i such that $H_i = Q_i^{-1} H_1 Q_i$. Thus, $\Delta_p^2 = \Pi\varphi(\delta_i) = \Pi Q_i^{-1} H_1^{\nu_i} Q_i$.

□

Remark. We can take any half-twist for H_1.

§4. Braid monodromy factorizations of Δ_p^2 and Hurwitz equivalence.

From Proposition 3.1 and 2.4, we know that a braid monodromy factorization

$$\Delta_p^2[D,K] = \prod_i \varphi(\delta_i)$$

is a factorization of $\Delta_p^2[D,K]$ in $B_p^+[D,K]$ induced from a g-base of $\pi_1(E \setminus N, u)$.

We define an equivalence relation on the set of B_p^+-factorizations of Δ_p^2. We start by classifying a Hurwitz move on $G \times \cdots \times G$ (G is a group) or on a set of factorizations.

Definition. Hurwitz moves R_k, R_k^{-1} on G^m:

Let $\vec{t} = (t_1, \dots, t_m) \in G^m$. We say that $\vec{s} = (s_1, \dots, s_m) \in G^m$ is obtained from $\vec{t}$ by the Hurwitz move R_k (or $\vec{t}$ is obtained from $\vec{s}$ by the Hurwitz move R_k^{-1}) if

$$s_i = t_i \quad \text{for } i \neq k\,,\ k+1\,,$$

$$s_k = t_k t_{k+1} t_k^{-1}\,,$$

$$s_{k+1} = t_k\,.$$

Definition. Hurwitz move on a factorization

Let G be a group $t \in G$. Let $t = t_1 \cdot \ldots \cdot t_m = s_1 \cdot \ldots \cdot s_m$ be two factorized expressions of t. We say that $s_1 \cdot \ldots \cdot s_m$ is obtained from $t_1 \cdot \ldots \cdot t_m$ by a Hurwitz move R_k if $(s_1, \dots, s_m)$ is obtained from $(t_1, \dots, t_m)$ by Hurwitz move R_k.

Definition. Hurwitz equivalence of factorization

Two factorizations are Hurwitz equivalent if they are obtained from each other by a finite sequence of Hurwitz moves.

In order to study equivalence relations, we first state that braid monodromy and Hurwitz moves are commutative.

Lemma 4.1. (Proof in [MoTe1], Chapter II). *Let D, K, u be as above.*

(a) *if $\Gamma_1, \ldots, \Gamma_n$ is a g–base of $\pi_1(D \setminus K, u)$ then $\Gamma_1 \cdot \ldots \cdot \Gamma_n$ is represented by the loop ∂D (taken with positive orientation).*

(b) *If $\{\Gamma_i'\}$ and $\{\Gamma_i\}$ are two g–bases of $\pi_1(D \setminus K, u)$ then each Γ_i' is conjugate to some Γ_{j_i}.*

(c) *By applying a Hurwitz move to a g–base, we get a g–base (see* Fig. 4.1*).*

(d) *Any two g–bases can be obtained from each other by a finite sequence of Hurwitz moves.*

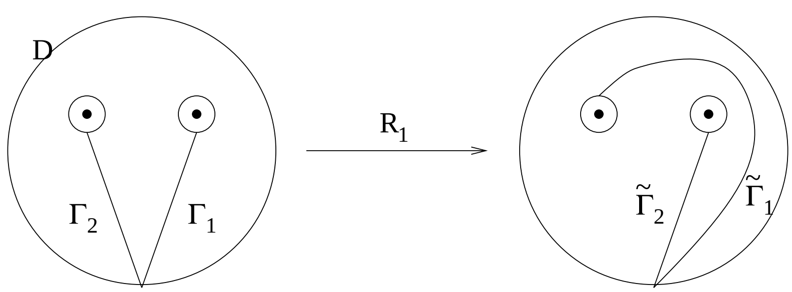

FIG. 4.1

Proposition 4.2. *Let B be a curve in $\mathbb{C}^2$, and let N, E, D, π, u be as above. Let φ be the braid monodromy of B with respect to $E \times D$, π, u. Let $n = \#N$. The following diagram*

$$\begin{array}{ccc} \{g\text{-}bases\ of\ \pi_1(E \setminus N, u)\} & \xrightarrow{\varphi^n} & (B_p[D,K])^n \\ {\scriptstyle R_k}\downarrow & & {\scriptstyle R_k}\downarrow \\ \{g\text{-}bases\ of\ \pi_1(E \setminus N, u)\} & \xrightarrow{\varphi^n} & (B_p[D,K])^n \end{array}$$

where R_k, is the k-th Hurwitz move, is commutative.

Proof. Follows immediately from definitions and the fact that

$$\varphi\left(\delta_k\delta_{k+1}\delta_k^{-1}\right) = \varphi(\delta_k)\varphi(\delta_{k+1})(\varphi(\delta_k))^{-1}. \quad \square$$

Different braid monodromy factorizations related to B that are derived from different g-bases are equivalent to each other:

Lemma 4.3. *Let B be a curve, let L, π, E, D, N, u, K be as in §3. Let φ be the braid monodromy of B with respect to L, u. Let $\Delta_p^2[D,K] = \prod\limits_i \varphi(\delta_i)$ and $\Delta_p^2[D,K] = \prod\limits_i \varphi(\delta_i')$ be two braid monodromy factorizations of $\Delta_p^2[D,K]$ corresponding to φ and two g-bases of $\pi_1(E \setminus N, u)$. Then the two factorizations are Hurwitz equivalent.*

Proof. Two g-bases $\{\delta_i\}$ and $\{\delta_i'\}$ of $\pi_1(E \setminus N, u)$ can be obtained from each other by a finite sequence of Hurwitz moves (Lemma 4.1). By Proposition 4.2 the same sequence of Hurwitz moves will transform $\{\varphi(\delta_i)\}$ into $\{\varphi(\delta_i')\}$. Thus the factorizations $\prod\limits_i \varphi(\delta_i)$ and $\prod\limits_i \varphi(\delta_i')$ are equivalent. $\square$

Lemma 4.4. *In the notation of the previous Lemma, if $\prod_i Z_i$ is Hurwitz equivalent to $\prod \varphi(\delta_i)$, then there exists a g-base $\{\delta_i'\}$ of $\pi_1(E \setminus N, u)$ such that $Z_i = \varphi(\delta_i')$.*

Proof. Let ε be the sequence of Hurwitz moves that takes $\varphi(\delta_i)$ to $\{Z_i\}$. Apply ε on $\{\delta_i\}$ to get $\{\delta_i'\}$. By Proposition 4.2, $\varphi(\delta_i') = Z_i$. $\square$

We conclude that:

Theorem 4.5. *Let B be a projective curve in $\mathbb{CP}^2$. Let $\varphi\colon \pi_1(E \setminus N, u) \to B_p[D,K]$ be its braid monodromy. The set of all braid monodromy factorizations of $\Delta_p^2[D,K]$ associated to B (presented by $\Delta_p^2 = \prod\limits_i \varphi(\delta_i)$, where $\{\delta_i\}$ is a g-bases of $\pi_1(E \setminus N, u)$) occupy a full equivalence class of factorizations of Δ_p^2 in B_p^+.*

Let B be an algebraic curve in $\mathbb{C}^2$ and let $\varphi\colon \pi_1(E \setminus N, u) \to B_p[D,K]$ be the braid monodromy of B defined by a braid monodromy factorization $\Delta_p^2 = \prod\limits_i \varphi(\delta_i)$,

where $\{\delta_i\}$ is a g-base of $\pi_1(E \setminus N, u)$). Acting on (D, K) by a diffeomorphism β, we obtain a new braid monodromy factorization $\Delta_p^2 = \prod_i \beta^{-1}\varphi(\delta_i)\beta$ associated to B.

Definition. Braid monodromy factorization type.

Two braid monodromy factorizations are called *equivalent with respect to Hurwitz moves and conjugations* if one of them can be obtained from the other by a finite sequence of Hurwitz moves, followed by a simultaneous conjugation by an element $\beta \in B_p$.

Two braid factorizations belong to the same *braid monodromy factorization type* if they are Hurwitz and conjugation equivalent.

§5. The centralizer of the multiple half-twists.

In this section we give a description of generators of the centralizer of the multiple half-twist which we will use in the proof of Theorem 1. We keep the same notations as in §1.

Theorem 5.1. *Let $(H_1, \dots, H_{p-1})$ be a frame of $B_p[D, K]$ given by a system of paths $\{\sigma_i\}_{i=1}^{p-1}$ and let $X = H_1^{\nu}$, $\nu \in \mathbb{N}$. Then the centralizer $C(X)$ of X in $B_p[D, K]$ is generated by $\Delta^2_{1,j}$, $j = 3, \dots, p$, and H_j, $j = 1, 3, \dots, p-1$.*

Proof. Denote by $C_1(X)$ a subgroup of $B_p[D, K]$ generated by H_1, $\Delta^2_{1,j}$, $j = 3, \dots, p$, and the set of all half-twists $H(\sigma)$ given by simple paths σ starting and ending at K and non-intersecting with σ_1. Let $C_2(X)$ be a subgroup generated by $\Delta^2_{1,j}$, $j = 3, \dots, p$, and H_j, $j = 1, 3, \dots, p-1$. It is clear that $C_2(X) \subset C_1(X) \subset C(X)$ and we must prove the inverse inclusions.

It is sufficient to prove Theorem 5.1 for even ν, since $C(X) \subset C(X^2)$.

Without loss of generality, we can assume that

$$D = \{\, (v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \le p^2 \,\}, \qquad K = \{q_0 = (0,0), \dots,\, q_{p-1} = (p-1, 0)\},$$

$$\sigma_i = [i-1, i] = \{\, (v_1, v_2) \in \mathbb{R}^2 \mid i-1 \le v_1 \le i,\ v_2 = 0 \,\}.$$

Denote by $\sigma_0 = [-p, 0] = \{\,(v_1, v_2) \in \mathbb{R}^2 \mid -p \le v_1 \le 0,\ v_2 = 0\,\}$ and $\sigma_p = [p-1, p] = \{\,(v_1, v_2) \in \mathbb{R}^2 \mid p-1 \le v_1 \le p,\ v_2 = 0\,\}$. We choose a point $u_0 \in \partial D$ such that u_0 does not lie in the lines $\{v_1 = i\}$, $i = 0, \dots, p-1$, and $\{v_2 = 0\}$. Consider an element $\gamma \in \pi_1(D \setminus K, u_0)$, $\gamma : [0,1] \to D \setminus K$, $\gamma(0) = \gamma(1) = u_0$. By slightly changing γ, we can assume that it is in a general position with respect to the lines $\{v_1 = i\}$, $i = 0, \dots, p-1$, $\{v_2 = 0\}$. The coordinates (v_1, v_2) define an orientation on D and on these lines. For each point $y \in \gamma \cap L$, where L is an oriented line in general position with respect to γ, the orientation chosen above allows us to define an intersection index $(\gamma, L)_y$ equal to ± 1. Let $t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1$ be a sequence of $t \in [0,1]$ for which $\gamma(t)$ belongs to one of the lines considered above. We associate a sequence $c(\gamma) = (a_0, \dots, a_n)$ of symbols $a_i \in \{u_0^{\pm 1}, o_0^{\pm 1}, \dots, o_p^{\pm 1}, h_0^{\pm 1}, \dots, h_{p-1}^{\pm 1}, l_0^{\pm 1}, \dots, l_{p-1}^{\pm 1}\}$ to this loop (a *code* of γ) as follows

(1) $a_0 = u_0$ and $a_n = u_0^{-1}$;

(2) $a_i = o_j^{\pm 1}$ if $\gamma(t_i) \in \sigma_j$ and the power coincides with $(\gamma, \sigma_j)_{\gamma(t_i)}$;

(3) $a_i = h_j^{\pm 1}$ if $\gamma(t_i) \in L_j$, where $L_j = \{v_1 = j, v_2 > 0\}$, and the power coincides with $(\gamma, L_j)_{\gamma(t_i)}$;

(4) $a_i = l_j^{\pm 1}$ if $\gamma(t_i) \in L_j$, where $L_j = \{v_1 = j, v_2 < 0\}$, and the power coincides with $(\gamma, L_j)_{\gamma(t_i)}$.

The code $c(\gamma) = (a_0, \dots, a_n)$ is said to be *reduced* if $a_i \ne a_{i+1}^{-1}$ for each i. We can associate the reduced code $c_r(\gamma)$ to a code $c(\gamma) = (a_0, \dots, a_n)$ removing from $(a_0, \dots, a_n)$ all pairs a_i, a_{i+1} such that $a_i = a_{i+1}^{-1}$.

We call $l_c(\gamma) = |c(\gamma)| = n + 1$ a *length* of $c(\gamma) = (a_0, \dots, a_n)$.

Similarly, to each path σ connecting points q_i and q_j in $D \setminus (K \setminus \{q_i, q_j\})$, we can associate a code $c(\sigma)$ if we add symbols $q_l^{\pm}$, $l = 1, \dots p$, to the symbols defined above, and we can define a notion of reduced code satisfying the following conditions:

(1) $a_1 = q_i$ and $a_n = q_j^{-1}$;

(2) $a_2 \in \{h_{i-1}^{-1}, l_{i-1}^{-1}, h_{i+1}, l_{i+1}\}$;

(3) $a_{n-1} \in \{h_{j-1}, l_{j-1}, h_{j+1}^{-1}, l_{j+1}^{-1}\}$;

(4) $a_l \neq a_{l+1}^{-1}$ for each pair a_l, a_{l+1} .

We define a sign $sgn(\sigma)$ of a code $c_r(\sigma)$ putting $sgn(\sigma) = 1$ if $a_2 \in \{h_{i-1}^{-1}, l_{i-1}^{-1}\}$ and $sgn(\sigma) = -1$ if $a_2 \in \{h_{i+1}, l_{i+1}\}$.

Lemma 5.2. *Loops γ_0 and γ_1 (resp. paths σ' and σ'') are homotopic in $D \setminus K$ if and only if $c_r(\gamma_1) = c_r(\gamma_2)$ (resp. $c_r(\sigma') = c_r(\sigma'')$).*

Proof. To each code $c(\gamma) = (a_0, \dots, a_n)$ we associate the element $a_0 \cdot \ldots \cdot a_n$ of a free group F generated by

$$u_0, o_0, \dots, o_p, h_0, \dots, h_{p-1}, l_0, \dots, l_{p-1}, q_1, \dots, q_p.$$

It is clear that a reduced word in F corresponds to a reduced code. It is well known that for each element in F the reduced word representing this element is uniquely defined. Therefore the reduced code is uniquely defined for each code.

It is clear that if γ_0 and γ_1 are homotopic in $D \setminus K$, then there exists a homotopy γ_s, $s \in [0,1]$, such that for almost all s except a finite number of $s \in \{s_1, \dots, s_k\}$, the loops γ_s are in general position with respect to the lines $\{v_1 = i\}$, $i = 0, \dots, p-1$, $\{v_2 = 0\}$, and for $s \in \{s_1, \dots, s_k\}$ the loops γ_s are touching one of these lines at one of the intersection points, and meet transversally at the other intersection points. Hence homotopic loops have the same reduced code. The inverse statement that if the reduced codes of homotopic loops γ_0 and γ_1 are equal to each other, then γ_0 and γ_1 are homotopic is evident.

The case of two paths is similar to the case of two loops considered above.

First, we will show that $C_1(X) = C(X)$. Consider an element $G \in C(X)$.

Lemma 5.3. *Let a diffeomorphism h be a representative of $G \in C(X)$. Then the*

simple paths $\sigma = h(\sigma_1)$ and σ_1 (considered as non-oriented paths) are homotopic in (D, K) (a homotopy leaving fixed K).

Proof. We have

$$X = G^{-1}XG = (G^{-1}H(\sigma_1)G)^{\nu} = H(h(\sigma_1))^{\nu} = H(\sigma)^{\nu},$$

i.e. $X = H_1^{\nu}$ can be represented as ν-th power of the half-twist defined by σ. Therefore Lemma 5.3 follows from

Lemma 5.4. *Let multiple full-twists $H(\sigma_1)^{\nu}$ and $H(\sigma)^{\nu}$ represent the same element in $B_p[D, K]$, where σ is a simple path. Then σ and σ_1 (considered as non-oriented paths) are homotopic in (D, K) (a homotopy leaving fixed K).*

Proof. Consider the reduced code $c_r(\sigma) = (q_i, a_2, \dots, a_{n-1}, q_j^{-1})$. If the length $|c_r(\sigma)| = 2$, then σ is homotopic to a path σ_l belonging to the chosen frame. Therefore, $l = 1$ follows from $H(\sigma_1)^{\nu} = H(\sigma_l)^{\nu}$.

Let us show that $|c_r(\sigma)| < 3$. In fact, assume that $|c_r(\sigma)| \geq 3$. Then for some $s \neq 0, 1$, there is a symbol $a_{l_0} \in \{h_s^{\pm 1}, l_s^{\pm 1}\}$ in the reduced code $c_r(\sigma)$. Let we have $a_{l_0} = h_s^{\varepsilon}$, $\varepsilon = \pm 1$ (the case $a_{l_0} = l_s^{\varepsilon}$ is similar and it will be omitted). We choose a point u_0 such that $s - 1 < v_1(u_0) < s$, $v_2(u_0) > 0$ and consider a loop $\gamma \in \pi_1(D \setminus K, u_0)$ whose reduced code $c_r(\gamma) = (u_0, o_{s-1}, l_s, o_s^{-1}, h_s^{-1}, u_0^{-1})$ (γ coincides with the $(p-s)$th element of a g-base). Then $(\gamma)H(\sigma_1)^{\nu} = \gamma$ and, therefore,

$$c_r((\gamma)H(\sigma_1)^{\nu}) = c_r(\gamma),$$

since we can choose a loop representing γ which does not intersect σ_1. To find a code $c((\gamma)H(\sigma)^{\nu})$, we associate a quadruple $b_{\sigma,\pm} = (b_1, b_2, b_3, b_4)$ to the pairs (q_i, a_2) and (a_{n-1}, q_j^{-1}) in the reduced code $c_r(\sigma)$ as follows

$$b_{(\sigma,+)} = \begin{cases} (o_{i-1}, l_i, o_i^{-1}, h_i^{-1}) & \text{if } a_2 = h_{i-1}^{-1}; \\ (l_i, o_i^{-1}, h_i^{-1}, o_{i-1}) & \text{if } a_2 = l_{i-1}^{-1}; \\ (o_i^{-1}, h_i^{-1}, o_{i-1}, l_i) & \text{if } a_2 = l_{i+1}; \\ (h_i^{-1}, o_{i-1}, l_i, o_i^{-1}) & \text{if } a_2 = h_{i+1} \end{cases}$$

and

$$b_{(\sigma,-)} = \begin{cases} (o_{j-1}, l_j, o_j^{-1}, h_j^{-1}) & \text{if } a_{n-1} = h_{j-1}; \\ (l_j, o_j^{-1}, h_j^{-1}, o_{j-1}) & \text{if } a_{n-1} = l_{j-1}; \\ (o_j^{-1}, h_j^{-1}, o_{j-1}, l_j) & \text{if } a_{n-1} = l_{j+1}^{-1}; \\ (h_j^{-1}, o_{j-1}, l_j, o_j^{-1}) & \text{if } a_{n-1} = h_{j+1}^{-1}. \end{cases}$$

We define

$$c_1c_2 = (a'_1, \dots, a'_{m_1}, a''_1, \dots, a''_{m_2})$$

for $c_1 = (a'_1, \dots, a'_{m_1})$ and $c_2 = (a''_1, \dots, a''_{m_2})$, and

$$c_1^{-1} = (a'^{-1}_{m_1}, \dots, a'^{-1}_1).$$

Put

$$c_r(\sigma)_{(k+)} = (a_{k+1}, \dots, a_{n-1}),$$

$$c_r(\sigma)_{(k-)} = (a_2, \dots, a_k),$$

$$r_\sigma = (a_2, , \dots, a_{n-1})b_{(\sigma,-)}(a_2, , \dots, a_{n-1})^{-1}b_{(\sigma,+)},$$

$$r_{\sigma,k} = c_r(\sigma)_{(k+)}b_{(\sigma,-)}c_r(\sigma)^{-1}_{(k+)}c_r(\sigma)^{-1}_{(k-)}b_{(\sigma,+)}c_r(\sigma)_{(k-)},$$

$$R_1(\sigma,k) = [(c_r(\sigma)_{(k+)}b_{(\sigma,-)}c_r(\sigma)^{-1}_{(k+)}),\ (c_r(\sigma)^{-1}_{(k-)}b_{(\sigma,+)}c_r(\sigma)_{(k-)})]$$

and

$$R_\nu(\sigma,k) = r^{\mu-1}_{\sigma,k}R_1(\sigma,k)r^{1-\mu}_{\sigma,k},$$

where $\nu = 2\mu$, $c_r(\sigma) = (a_1, a_2, \dots, a_{n-1}, a_n)$, $[c_1, c_2] = c_1c_2c_1^{-1}c_2^{-1}$, and each code $r_{\sigma,k}$ can be obtained from r_σ by means of a cyclic permutation. It is clear that $r_{\sigma,k}$ and $R_\nu(\sigma,k)$ are reduced codes.

Without loss of generality, we can choose σ and γ such that $c(\sigma) = c_r(\sigma)$ and $\gamma = \delta \circ h_s \circ C_s \circ h_s^{-1} \circ \delta^{-1}$, where C_s is a circle of a small radius with the center at q_s, h_s is a path along the line $v_1 = s$, and δ is the shortest path along ∂D connecting u_0 and the intersection point of ∂D and the ray $\{v_1 = s,\ v_2 > 0\}$. Let $w_1 > w_2 > \cdots > w_m > 0$ be the sequence of values of v_2 corresponding to the

intersection points of the ray $\{v_1 = s,\ v_2 > 0\}$ and σ, and let $k_1, k_2, \dots, k_m$ be the indices of a_{k_l} in $c_r(\sigma)$ and corresponding to these values of v_2 (by definition, these a_{k_l} are equal to $h_s^{\varepsilon_{k_l}},\ \varepsilon_{k_l} = \pm 1$). Then

$$c((\gamma)H(\sigma)^\nu) =$$

$$= (u_0, \prod_{l=1}^{m} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}}, o_{s-1}, l_s, o_s^{-1}, h_s^{-1}, (\prod_{l=1}^{m} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}})^{-1}, u_0^{-1})$$

if q_s is not the starting point of σ. If q_s is the starting point of σ, then the code $c((\gamma)H(\sigma)^\nu)$ is equal to

$$(u_0, \prod_{l=1}^{m-1} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}}, \delta_0(\sigma), (\prod_{l=1}^{m-1} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}})^{-1}, u_0^{-1})$$

if $a_2 = h_{s-1}^{-1}$, where

$$\delta_0(\sigma) = r_\sigma^{\mu-1}\delta_0'(\sigma)r_\sigma^{1-\mu},$$

and

$$\delta_0'(\sigma) =$$

$$= (a_2, \dots, a_{n-1})b_{(\sigma-)}(a_2, \dots, a_{n-1})^{-1}b_{(\sigma+)}(a_2, \dots, a_{n-1})b_{(\sigma-)}^{-1}(a_2, \dots, a_{n-1})^{-1}),$$

and $c((\gamma)H(\sigma)^\nu)$ is equal to

$$(u_0, \prod_{l=1}^{m} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}}, \delta(\sigma), (\prod_{l=1}^{m} R_\nu(\sigma, k_l)^{-sgn(\sigma)\varepsilon_{k_l}})^{-1}, u_0^{-1})$$

if $a_2 \neq h_{s-1}^{-1}$, where

$$\delta = \begin{cases} (o_{s-1}, \delta_0(\sigma), o_{s-1}^{-1}) & \text{if } a_2 = l_{s-1}^{-1}, \\ (o_{s-1}, l_s, \delta_0(\sigma), l_s^{-1}, o_{s-1}^{-1}) & \text{if } a_2 = l_{s+1}, \\ (o_{s-1}, l_s, o_s^{-1}\delta_0(\sigma), o_s, l_s^{-1}, o_{s-1}^{-1}) & \text{if } a_2 = h_{s+1}. \end{cases}$$

Without loss of generality, we can assume that $\nu \gg 2$. In addition, it is easy to see that $c_r(r_{\sigma,k_i} r_{\sigma,k_j}^{-1}) \neq \emptyset$ for $k_i \neq k_j$, since $c_r(\sigma)$ is the reduced code. Therefore, $c_r((\gamma)H(\sigma)^\nu) \neq c_r(\gamma) = (u_0, o_{s-1}, l_s, o_s^{-1}, h_s^{-1}, u_0^{-1})$. Lemma 5.4 is proved.

The diffeomorphism h representing the element $G \in C(X) \subset B_p[D,K]$ is defined up to isotopy, hence by Lemma 5.4 we can assume that $h(\sigma_1)=\sigma_1$, $h(q_i)=q_i$ for $i=1,\,2$.

Let us show that there exists a diffeomorphism g_2 representing an element $G_2 \in C_1(X)$ such that $g_2 \circ h$ leaves fixed the paths σ_1 and σ_2. In fact, consider the path $h(\sigma_2)$. The point q_2 is the starting point of this path and some point $q_r \neq q_1, q_2$ is the end point. Let $s:[0,1]\to D$ be a parametrization of this path such that $s(0)=q_r$. Consider the reduced code $c_r(h(\sigma_2))=(a_1,\dots,a_n)$ of the path $h(\sigma_2)$, $a_1=q_r$, $a_n=q_2^{-1}$. Let for some i the symbol a_i be equal to $o_j^{\pm 1}$, where $1<j<p$. We choose among all such i an index i_0 such that $a_{i_0}=o_{j_0}^{\pm 1}$ for which the following conditions are fulfiled:

(i) if $j_0=2$, then there is no other intersection point of $h(\sigma_2)$ and σ_2 lying in σ_2 between $s(t_{i_0})$ and q_3 ;

(ii) if $j_0=r$, then there is no other intersection point of $h(\sigma_2)$ and σ_r lying in σ_r between $s(t_{i_0})$ and q_r ;

(iii) if $j_0=r+1$, then there is no other intersection point of $h(\sigma_2)$ σ_{r+1} lying in σ_{r+1} between $s(t_{i_0})$ and q_{r+1};

(iv) if $j_0\neq 2,r,r+1$, then there is no other intersection point of σ_{j_0} and $h(\sigma_2)$ lying in σ_{j_0} between either q_{j_0} and $s(t_{i_0})$ or $s(t_{i_0})$ and q_{j_0+1}.

Consider one of these cases (the other cases are similar). For instance, let $j_0\neq 2,r,r+1$ and assume that there is no intersection point of σ_{j_0} and $h(\sigma_2)$ lying on σ_{j_0} between q_{j_0} and $s(t_{i_0})$. Denote by $\widetilde{h(\sigma_2)}$ a path consisting of a part of $h(\sigma_2)$ starting at q_r and ending at $s(t_{i_0})$ and a part of σ_{j_0} starting at $s(t_{i_0})$ and ending at q_{j_0}. Let us choose a smooth path $\widetilde{\sigma}$, sufficiently closed to $\widetilde{h(\sigma_2)}$ such that $\widetilde{\sigma}$ connects q_r and q_{j_0}, $\widetilde{h(\sigma_2)}\cap\widetilde{\sigma}=\{q_r,q_{j_0}\}$, and such that if we move along $\widetilde{\sigma}$ starting at q_r, then the path $\widetilde{h(\sigma_2)}$ is situated to the right from $\widetilde{\sigma}$. Perform a half-twist $H(\widetilde{\sigma})\in C_1(X)$. Let a diffeomorphism h_1 be a representative of $H(\widetilde{\sigma})$. It is easy to see that $h_1(h(\sigma_2))$ is

isotopic to the path having the following code $c(h_1 \circ h) = (\widetilde{a}_0, \dots, \widetilde{a}_{n-i_0-l})$, where l is a non-negative integer, $\widetilde{a}_0 = q_{j_0}$, $\widetilde{a}_j = a_{j+i_0+l}$.

Denoting again by h the diffeomorphism $h_1 \circ h$ and repeating the process described above, we can assume that the code $c(h(\sigma_2)) = (a_0, \dots, a_n)$ of the curve $h(\sigma_2)$ satisfies the condition: for any i the symbol $a_i \neq o_j^{\pm 1}$, where $0 < j < p$. We can assume for definiteness that $a_1 = l_{r+1}$ and $a_{n-1} = l_0$ (the other cases are similar). In this case, it it is easy to see that

$$c(h) = (q_r, l_{r+1} \dots, l_{p-1}, o_p^{-1}, h_{p-1}^{-1}, \dots, h_1^{-1}, o^0, l_0, q_2^{-1}).$$

It is easy to check that the path $(h(\sigma_2))\Delta_{r,p} \circ \Delta_{3,p} \circ \Delta^2_{1,2}$ is isotopic to σ_2. But $\Delta_{r,p} \circ \Delta_{3,p} \circ \Delta^2_{1,2} \in C_2(X) \subset C_1(X)$. Thus, multiplying G by $\Delta_{r,p} \circ \Delta_{3,p} \circ \Delta^2_{1,2}$, we can assume that the diffeomorphism h representing $G \in C(X)$ leaves fixed the path $L_{1,3}$.

Repeating the stated above consecutively for $\sigma_3, \dots, \sigma_{p-1}, \sigma_0$, we can assume that the diffeomorphism h representing $G \in C(X)$ leaves fixed the path $L_{0,p} = \sigma_0 \cup L_{1,p}$. In this case considering the code of the path $h(\sigma_p)$, it is easy to see that $h(\sigma_p)$ is isotopic to σ_p in (D, K). Therefore we can assume that h leaves fixed the diameter $L_{0,p+1} = \sigma_0 \cup L_{1,p} \cup \sigma_p$. But in this case, h is a representative of the identity element of $B_p[D, K]$. Thus, $C_1(X) = C(X)$.

To prove that $C_1(X) = C_2(X)$, it is sufficient to show that if a half-twist $H(\sigma)$ is given by a simple path σ starting and ending at K and non-intersecting with σ_1, then $H(\sigma) \in C_2(X)$.

We note that, for any $G \in B_p[D, K]$, the element $G^{-1}H(\sigma)G$ is a half-twist given by the path $(\sigma)G$. Therefore, to prove that any such half-twist $H(\sigma) \in C_2(X)$, one repeats the arguments stated above using induction on $l_c(\sigma)$.

§6. Smooth isotopy of fiber space.

Let M be a smooth variety. By definition, a diffeomorphism $F : M \times [0, 1] \rightarrow$

$M \times [0,1]$ (or simply $F_t : M \to M$) is a smooth isotopy if

(1) $F(M \times \{t\}) = M \times \{t\}$ for all $t \in [0,1]$;

(2) $F_0 = F_{|M\times\{0\}} : M \times \{0\} \to M \times \{0\}$ is the identity map.

Without loss of generality, we shall assume that the isotopy F_t satisfies an additional condition

(3) $F_t = F_0$ if $t \leq \varepsilon$ and $F_t = F_1$ if $t \geq 1-\varepsilon$ for some $\varepsilon > 0$.

Indeed, instead of a smooth isotopy F_t, we can consider a smooth isotopy $\widetilde{F}_t = F_{h(t)}$, where $h : [0,1] \to [0,1]$ is a smooth monotone function such that $h(t) = 0$ if $t \leq \varepsilon$ and $h(t) = 1$ if $t \geq 1-\varepsilon$.

By definition, the composition of smooth isotopies F'_t and F''_t is the smooth isotopy $F_t = F''_t \circ F'_t$ given by $F_t = F'_{2t}$ if $t \leq \frac{1}{2}$ and $F_t = F''_{2t-1} \circ F'_1$ if $t \geq \frac{1}{2}$.

Let U be a neighbourhood in M and ∂U its boundary. Let $F_t : \overline{U} \to \overline{U}$ be a smooth isotopy. It is evident that if the restriction of $F_t : \overline{U} \to \overline{U}$ to a neighbourhood of ∂U is the identity map for all $t \in [0,1]$, then F_t can be extended to a smooth isotopy $\widetilde{F}_t : M \to M$ such that $\widetilde{F}_{t|M\setminus U}$ is the identity map for all t.

Let B_1 and B_2 be two plane curves. As in Section 1, let

$K_i(x) = \{y \bigm| (x,y) \in B_i\}$, $i = 1,2$, ($K_i(x) =$ projection to y-axis of $\pi^{-1}(x) \cap B_i$),

$$N_i = \{x \bigm| \#K_i(x) \lneqq p\}.$$

$$M'_i = \{(x,y) \in B_i \bigm| \pi\bigm|_{B_i} \text{ is not étale at } (x,y)\}\ (\pi(M'_i) = N_i).$$

Let E_R (resp. D_R) be a closed disk of radius R with center at the origin o on x-axis (resp. y-axis) such that $M'_i \subset E_R \times D_R$, $N_i \subset \mathrm{Int}(E)$.

Assume that $\#K_i(o) = p$.

For each $u_{i,j} \in N_i$, $j = 1,\dots,n$, we choose a disc $E_{i,j}$ of small radius $\epsilon \ll 1$ with center at $u_{i,j}$ and choose simple paths $T_{i,1},\dots,T_{i,n}$ connecting $u_{i,j}$ with o such

that $< T_{i,1}, \dots, T_{i,n} >$ is a bush. For each $j = 1, \dots, n$ we choose a small tubular neighbourhood $U_{i.j}$ of $T_{i,j}$ and choose a disc E_o of radius $\epsilon \ll 1$ with center at o such that

(1) $U_{i,j_1} \cap U_{i,j_2} \subset E_o$ for $j_1 \neq j_2$;

(2) $E_{i,j} \cap E_o = \emptyset$ for all j ;

(3) the set

$$U_{\Gamma_i} = (\bigcup_{j=1}^{n} U_{i,j}) \cup (\bigcup_{j=1}^{n} E_{i,j}) \cup E_o$$

is diffeomorphic to a disc and its boundary ∂U_{Γ_i} is a smooth simple loop.

Such U_{Γ_i} will be called *a tubular neighbourhood* of a g-base $\Gamma_i = (l(T_{i,1}), \dots, l(T_{i,n}))$ of the fundamental group $\pi_1(E_R \setminus N_i, o)$.

The following lemmas are well-known.

Lemma 6.1. *In the notation described above, for $N_i \subset E_R$, $i = 1, 2$, let U_{Γ_i} be two tubular neighbourhoods of g-bases Γ_i of $\pi_1(E_R \setminus K_i, o)$. Assume that $\#N_1 = \#N_2$. Then there exists a smooth isotopy $f_t : E_R \to E_R$, $t \in [0, 1]$, such that*

(1) *f_t is the identity map in a neighbourhood of the boundary of E_R ;*

(2) *$f_t(o) = o$ for all $t \in [0, 1]$;*

(3) *$f_1(E_{1,j}) = E_{2,j}$ for each $j = 1, \dots, n$.*

(4) *$f_1(U_{\Gamma_1}) = U_{\Gamma_2}$.*

Lemma 6.2. *The isotopy f_t from Lemma 6.1 can be extended to a smooth isotopy $F_t : E_R \times \mathbb{C}^1 \to E_R \times \mathbb{C}^1$ such that*

(1) *$F_{t|E_R \times D_R} = f_t \times Id$;*

(2) *F_t is the identity map outside $E_R \times D_{2R}$.*

Lemma 6.3. *Let smooth real functions $\alpha(u, v), \beta(u, v)$ satisfy the following inequalities*

$$\varepsilon_1 + \varepsilon_2 < \alpha(u, v) < 1 - \varepsilon_1 - \varepsilon_2;$$

$$\varepsilon_1 + \varepsilon_2 < \beta(u, v) < 1 - \varepsilon_1 - \varepsilon_2$$

for all $(u, v) \in E_1 = \{\sqrt{u^2 + v^2} \leq 1\}$ *and for some positive* $\varepsilon_1, \varepsilon_2 \ll 1$. *Then there exists a smooth real function* $f_{(\alpha,\beta)t}(z, u, v)$, $(z, t, u, v) \in [0, 1] \times [0, 1] \times E_1$, *satisfying the following conditions:*

(1) $f_t(z, u, v)$ *is a monotone function for each fixed* $(t, u, v) \in [0, 1] \times E_1$ *;*

(2) $f_t(z, u, v) \equiv z$ *if* $0 \leq z \leq \varepsilon_1$*;*

(3) $f_t(z, u, v) \equiv z$ *if* $1 - \varepsilon_1 \leq z \leq 1$*;*

(4) $f_t(z, u, v) = z + t(\beta(u, v) - \alpha(u, v))$ *if* $\alpha(u, v) - \varepsilon_2 \leq z \leq \alpha(u, v) + \varepsilon_2$.

Proposition 6.4. *Let* $\mathcal{B} = (b_1(x), \dots, b_p(x))$ *be a collection of non-intersecting sections of the projection* $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid |x| \leq 1\}$. *Then there exists a smooth isotopy* $F_t : E \times D_R \to E \times D_R$ *such that*

(1) $F_t(x, y) = (x, F_{t,x}(y))$ *for all* t *and* x *;*

(2) $F_1(\mathcal{B})$ *is a collection of constant sections.*

Moreover, if all $b_j(x)$ *are constant sections (equal to* b_j*) over a neighbourhood of the boundary of* E, *then the isotopy* F_t *can be chosen in such a way that*

(3) F_t *is the identity map in the neighbourhood of the boundary of* $E \times D_R$ *for* $t \in [0, 1]$ *;*

(4) $F_1(x, b_j(x)) = (x, b_j)$.

Proof. The Proposition follows from the the following Lemma.

Lemma 6.5. *Let* $\mathcal{B} = (b_1(x), b_2, \dots, b_k)$ *be a collection of non-intersecting sections of the projection* $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid |x| \leq 1\}$ *such that* $b_2, \dots, b_k$ *are constant sections. Then there exists a smooth isotopy* $F_t : E \times D_R \to E \times D_R$ *such that*

(1) $F_t(x, y) = (x, F_{t,x}(y))$ *for all* t *and* x *;*

(2) $F_t(x, b_j) = (x, b_j)$ *for all* t *and* j *;*

(3) *$F_1(x, b_1(x))$ is a constant section.*

Moreover, if $b_1(x)$ is a constant section (equal to b_1) over a neighbourhood of the boundary of E, then the isotopy F_t can be chosen in such a way that

(4) *F_t is the identity map in a neighbourhood of the boundary of $E \times D_R$ for $t \in [0,1]$;*

(5) *$F_1(x, b_j(x)) = (x, b_j)$.*

Proof. Consider the set $S = \{y \in D_R \mid y = b_1(x),\, x \in E\}$. At first, assume that we can find a simply connected neighbourhood $U \subset D_R$ of S such that $b_j \notin U$ for $j = 2, \dots, k$. By Riemann's Theorem, there exists a complex-analytic bijective morphism $\varphi : U \to U_1$, where $U_1 = \{\, z = z_1 + iz_2 \in \mathbb{C} \mid 0 < z_1 < 1,\ 0 < z_2 < 1, \}$. Since S is a compactum, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\varepsilon_1 + \varepsilon_2 < z_l < 1 - \varepsilon_1 - \varepsilon_2, \qquad l = 1, 2,$$

for $z_1 + iz_2 \in \varphi(S)$. We have two smooth functions $\alpha_1(x),\ \alpha_2(x)$, where $\varphi(b_1(x)) = \alpha_1(x) + i\alpha_2(x)$, satisfying

$$\varepsilon_1 + \varepsilon_2 < \alpha_l(x) < 1 - \varepsilon_1 - \varepsilon_2.$$

By Lemma 6.3, for any two smooth functions $\beta_1(x)$ and $\beta_2(x)$ (for instance, if $\beta_1(x)$ and $\beta_2(x)$ are constant functions such that $\varphi(b_1) = \beta_1 + i\beta_2$) satisfying

$$\varepsilon_1 + \varepsilon_2 < \beta_l(x) < 1 - \varepsilon_1 - \varepsilon_2, \qquad l = 1, 2,$$

there exists a smooth isotopy $F_t : E \times U_1 \to E \times U_1$ given by

$$F_t^*(x) \equiv x\,;$$

$$F_t^*(z_1) = f_{(\alpha_1, \beta_1) th(|x|)}(z_1, u, v)\,;$$

$$F_t^*(z_2) = f_{(\alpha_2, \beta_2) th(|x|)}(z_2, u, v)\,,$$

where $u+iv=x$, $f_{(\alpha_l,\beta_l)t}(z_l,u,v)$ is a function described in Lemma 6.3, and $h(r)$ is a smooth monotone real function such that $h(r)=1$ if $r<\varepsilon$ and $h(r)=0$ if $r>1-\varepsilon$. It is easy to check that the constructed isotopy F_t satisfies properties (1) - (4) of Lemma 6.5.

If S is not "wound" around every point b_j, $j=2,\dots,k$, then we can find such a simply connected neighbourhood U of S. Otherwise we will show that there exists a sequence of isotopies which "unwinds" S. To show this, we fix the point $x_0=1\in E$. Let $\gamma(t)=x_t$ be a smooth simple (without self-intersections) path joining x_0 with a point $x_1\in E$. Denote by $n_{j,\gamma}(x_1)$ the integral part of the number of rotations around the point b_j of a point moving along the path $b_1(\gamma(t))$. Evidently, $n_j(x_1)=n_{j,\gamma}(x_1)$ depends only on the point x_1, and not on γ joining x_0 and x_1, since E is simply connected and $b_1(x)$ is a smooth function. Denote by

$$E_{n_2,\dots,n_k}=\{\, x\in E \mid n_j(x)=n_j,\ j=2,\dots,k\,\}$$

and let $E_{\max}$ (resp. $E_{\min}$) be the set $E_{n_2^0,\dots,n_k^0}$, where n_j^0 is a local maximum (resp. minimum) of $n_j(x)$ for all j. Evidently, if $E_{\max}=E_{\min}=E_{0,\dots,0}$, then we can find a simply connected neibourhood U of S such that $b_j\notin U$ for $j=2,\dots,k$.

Denote by $\bar l_j$ the line passing through b_1 and b_j and choose a coordinate u_j in $\bar l_j$ such that $u_j(b_1)=0$, where $b_1=b_1(x_0)$. Let $l_j=A_jy_1+B_jy_2$ be a linear function such that $\bar l_j=\{\, l_j(y_1,y_2)=0\,\}$, where $y_1=\operatorname{Re} y$ and $y_2=\operatorname{Im} y$.

Let

$$S_j=S\cap\{\, y\in\bar l_j \mid u_j(y)<u_j(b_j) \text{ if } u_j(b_j)>0 \text{ and } u_j(y)>u_j(b_j) \text{ if } u_j(b_j)<0\,\}.$$

Denote by $C_j=\{\, x\in E \mid y=b_1(x)\in S_j\,\}$. Consider $G=E\setminus\bigcup_{j=2}^{k}C_j$ which is a disjoint union of a finite number of connected components. Any two neighbouring G_m and G_l are separated by the connected component C_j^0 of C_j for some j. If $n_j(m)=n_j(l)$, where $n_j(m)=n_j(x)$ for $x\in G_m$, then we change G_m and G_l to the union $G_m\cup G_l\cup C_j^0$.

Evidently, there exists a connected component G^0 of G whose boundary ∂G^0 is connected and therefore, G^0 is simply connected. Moreover, it is easy to see that there exists a simply connected neighbourhood U_0 of $\{ y \in D_R \mid y = b_1(x), x \in \overline{G}^0 \}$ such that $b_j \notin U_0$ for $j = 2, \dots, k$. There is j_0 such that ∂G^0 is a subset of $\{ l_{j_0}(b_1(x)) = 0 \}$. By the choice of G^0, we can assume that $l_{j_0}(b_1(x)) \geq 0$ for all $x \in G^0$. Since $l_{j_0}(b_1(x))$ is a smooth function, then the set $\{ l_{j_0}(b_1(x)) = \delta \}$ is a smooth curve for δ close to 0. Therefore, without loss of generality, we can change G^0 by a set $\widetilde{G}^0$ such that

(1) G^0 is closed to $\widetilde{G}^0$ and contained in $\widetilde{G}^0$;

(2) the boundary of $\widetilde{G}^0$ is the subset of $\{ l_{j_0}(b_1(x)) = \delta \}$ for some $\delta < 0$ and close to 0 ;

(3) the closure of $b_1(\widetilde{G}^0)$ is contained in U_0, where U_0 is a simply connected open set such that $b_j \notin U_0$ for $j = 2, \dots, k$.

As above, we can find holomorphic coordinate $z = z_1 + iz_2$, $z = \varphi(y)$, in U_0 such that $U_0 \simeq \{ 0 < z_1 < 1 \} \times \{ 0 < z_1 < 1 \}$. Evidently, for the closure G^1 of $\widetilde{G}^0$ there exist smooth functions $\alpha : G^1 \to U_0$, $z = \varphi(b_1(x)) = \alpha_1(x) + i\alpha_2(x)$, and $\beta(x) = \beta_1(x) + i\beta_2(x)$ such that $\beta(x) = \varphi(b_1(x))$ for x lying in a neighbourhood of ∂G^1 and $l_{j_0}(\beta(x)) < 0$ for all $x \in G^1$. As above applying Lemma 6.3, we can find a smooth isotopy $F_t : G^1 \times U_0 \to G^1 \times U_0$ such that $F_1(x, b_1(x))$ does not meet the set $\{(x, y) \mid x \in G^1, y \in \cup \bar{l}_j\}$ and such that F_t is the identity map in a neighbourhood of the boundary of $G^1 \times U_0$. Hence this isotopy can be extended to the isotopy of $E \times D_R$. Evidently, for the image $F_1(b_1(x))$ of the section $b_1(x)$, we can repeat the construction of the set G and observe that the number of connected components of G is decreased. Hence, after several similar steps, we construct the desired isotopy as the composition of isotopies constructed in each step.

From the consideration described above, the following remark follows.

Remark 6.6. *Let $C \subset D$ be a connected set such that the closure $\overline{C}$ and the*

boundary ∂D of D have a non-empty intersection. Assume that $b_1(x) \equiv b_1$ for $x \in C$. Then on each step (except the last one), G^0 can be chosen in such a way that $C \not\subset G^0$.

The proof of Lemma 6.5 in the case when $b_1(x)$ is a constant section over a neighbourhood of the boundary of D follows from Remark 6.6. $\square$ for Lemma 6.5

Remark 6.7. *Let $\mathcal{B}_1 = (b_{1,1}(x), \dots, b_{1,p}(x))$ and $\mathcal{B}_2 = (b_{2,1}(x), \dots, b_{2,p}(x))$ be two collections of non-intersecting sections of the projection $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid |x| \le 1\}$ such that for all j the sections $b_{1,j}(x)$ and $b_{2,j}(x)$ coincide over a neighbourhood U of the boundary of E. It is easy to see that the smooth isotopies $F'_t : E \times D_R \to E \times D_R$ for $\mathcal{B}_1$ and $F''_t : E \times D_R \to E \times D_R$ for $\mathcal{B}_2$ from Proposition 6.4 can be chosen in such a way that F'_t and F''_t coincide over U.*

Let $\mathcal{B} = (b_1(x), \dots, b_p(x))$ be a collection of non-intersecting sections of the projection $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid |x| \le 1\}$. Denote by

$$U_\varepsilon(b_j(x)) = \{(x, y) \in D \times E_R \mid |y - b_j(x)| < \varepsilon\}$$

a tubular neighbourhood of the section $b_j(x)$ and put $U_\varepsilon(\mathcal{B}) = \cup U_\varepsilon(b_j(x))$. Let $F_t : E \times D_R \to E \times D_R$ be a smooth isotopy. Consider a smooth map $\widetilde{F}'_t : U_\varepsilon(\mathcal{B}) \times [0,1] \to E \times D_R \times [0,1]$ given by

$$\widetilde{F}'_t(x, y) = (x, y + (F_t(b_j(x)) - b_j(x))$$

if $(x, y) \in U_\varepsilon(b_j(x))$. We observe that $\widetilde{F}'_{t|\mathcal{B}} = F_{t|\mathcal{B}}$.

Using the standard technique of pasting together vector fields, one can prove the following

Lemma 6.8. *Let an isotopy $F_t : E \times D_R \to E \times D_R$ have properties $(1) - (4)$ of Proposition 6.4. Then for some $\varepsilon_1 \ll \varepsilon$ the map $\widetilde{F}'_t : U_\varepsilon(\mathcal{B}) \times [0,1] \to E \times D_R \times [0,1]$ can be extended to an isotopy $\widetilde{F}_t : E \times D_R \to E \times D_R$ having properties $(1) - (4)$ of Proposition 6.4.*

Lemma 6.9. *Let $\mathcal{B}_i = (b_{i,1}(x), \dots, b_{i,p}(x))$ be two collections of non-intersecting sections of the projection $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid |x| \leq 1\}$. Then there exists a smooth isotopy $F_t : E \times D_R \to E \times D_R$ such that*

(1) *$F_t(x,y) = (x, F_{t,x}(y))$ for all t and x;*
(2) *for each j there exists a neighbourhood U_j of the section $b_j(x)$ such that $F_{1|U_j}$ is holomorphic in y;*
(3) *$F_1(\mathcal{B}_1) = \mathcal{B}_2$ over $E_{R_1} \subset E$ for some $R_1 < 1$.*
(4) *F_t is the identity map in the neighbourhood of the boundary of $E \times D_R$ for $t \in [0,1]$.*

Proof. By Proposition 6.4 and Lemma 6.8, there exists a smooth isotopy $\widetilde{F}_t : E \times D_R \to E \times D_R$ having properties (1) - (2) of Lemma 6.9 and such that $\widetilde{F}_1(\mathcal{B}_1) = \mathcal{B}_2$. Let $h(r)$ be a smooth monotone function such that $h(r) = 1$ if $r \leq R_1 < 1 - \varepsilon$ and $h(r) = 0$ if $r \geq 1 - \varepsilon$. Then $F_t(x,y) = \widetilde{F}_{h(|x|)t}(x,y)$ has all properties of Lemma 6.9.

Lemma 6.10. *Let $\mathcal{B}_i = (b_{i,1}(x), \dots, b_{i,p}(x))$ be two collections of non-intersecting sections of the projection $\pi : E \times D_R \to E = \{x \in \mathbb{C}^1 \mid x_1 = Re\, x \in [0,1],\ x_2 = Im\, x \in [0,1]\}$ such that over $E_\varepsilon = \{0 \leq x_1 \leq \varepsilon\} \times \{0 \leq x_2 \leq 1\} \cap \{1 - \varepsilon \leq x_1 \leq 1\} \times \{0 \leq x_2 \leq 1\}$ the sections $(b_{1,1}(x), \dots, b_{1,p}(x)) = (b_{2,1}(x), \dots, b_{2,p}(x)) = (b_1, \dots, b_p)$ are coinciding constant sections. Let the geometric braids*

$$\overline{\mathcal{B}}_1 = (b_{1,1}(x_1), \dots, b_{1,p}(x_1)) \ \ and \ \ \overline{\mathcal{B}}_2 = (b_{2,1}(x_1), \dots, b_{2,p}(x_1)), \quad x_1 \in [0,1],$$

are two representatives of the same element in the braid group B_p. Then there exists a smooth isotopy $F_t : E \times D_R \to E \times D_R$ such that

(1) *$F_t(x,y) = (x, F_{t,x}(y))$ for all t and x;*
(2) *for each j there exists a neighbourhood U_j of the section $b_j(x)$ such that $F_{1|U_j}$ is holomorphic in y;*

(3) *$F_1(\mathcal{B}_1) = \mathcal{B}_2$ over $\{0 \le x_1 \le 1\} \times \{\, |x_2| < \varepsilon_2 \,\}$ for some $\varepsilon_2 > 0$.*

(4) *F_1 is the identity map in the neighbourhood of the boundary of $E \times D_R$.*

Proof. By Proposition 6.4, there exists a smooth isotopy $\widetilde{F}_t$ such that $\widetilde{F}_1(b_{1,j}(x)) = b_j$, $j = 1, \dots, p$, are constant sections.

Fix the point $x_0 = (0,0)$. As above, for each section $\widetilde{F}_1(b_{2,j_0}(x))$ we can define a function $n_{j_0,j}(x)$ equals to the number of rotations around b_j. Evidently, $\widetilde{F}_1(\overline{\mathcal{B}}_1)$ and $\widetilde{F}_1(\overline{\mathcal{B}}_2)$ are also the representatives of the same element in B_p. Therefore, for each section $\widetilde{F}_1(b_{2,j_0}(x))$ the number of rotations $n_{j_0,j}(x) = 0$ for $x \in \{1-\varepsilon \le x_1 \le 1\} \times \{0 \le x_2 \le 1\}$. Therefore, Lemma 6.10 follows from Remark 6.6 and Lemma 6.9.

§7. Braid monodromy factorization types and diffeomorphisms of pairs.

Consider a linear projection $\pi : \mathbb{CP}^2 \to \mathbb{CP}^1$ with center at $z \in \mathbb{CP}^2$.

Definition. Semi-algebraic curve (with respect to π)

A closed subset $B \subset \mathbb{CP}^2$, $z \notin B$, is called a semi-algebraic curve with respect to π, if for each point $x \in B$ there exist a neighbourhood $U \subset \mathbb{CP}^2$ of x and local analytic coordinates (z_1, z_2) in U such that

(1) $\pi_{|U}$ is given by $\pi(z_1, z_2) = z_1$;

(2) either $B \cap U$ is a smooth section of $\pi_{|U}$ over $\pi(U)$, or $B \cap U$ coincides with a set given by the equation $f(z_1, z_2) = 0$, where $f(z_1, z_2)$ is an analytic function.

A semi-algebraic curve is (generalized) *cuspidal* if in (2) the function f coincides with $f = z_1^k - z_2^2$, $k \in \mathbb{N}$. The point $(0,0)$ in a neighbourhood of which B is given by $z_1^k - z_2^2 = 0$ is called a *singular* point of B.

Obviously, any algebraic curve $B \subset \mathbb{CP}^2$ is a semi-algebraic curve with respect to a generic projection.

As in the algebraic case, we can resolve the singular points of a semi-algebraic curve B by means of a composition of σ-processes $\nu : \overline{\mathbb{P}}^2 \to \mathbb{P}^2$ (we needn't resolve the tangent points of B with the fibers of π) and obtain a non-singular Rimannian surface $\overline{B} \subset \overline{\mathbb{P}}^2$. The composition $(\pi \circ \nu)_{|\overline{B}} : \overline{B} \to \mathbb{P}^1$ allows us to introduce a complex structure on $\overline{B}$ such that $(\pi \circ \nu)_{|\overline{B}}$ is a holomorphic map, but $\nu_{|\overline{B}} : \overline{B} \to \mathbb{P}^2$ is not holomorphic and it is a C^∞-map only.

It is clear that as in the algebraic case one can define the braid monodromy with respect to π, braid monodromy factorization, and braid monodromy factorization type for any semi-algebraic curve.

Let $\sigma : \mathbb{F}_1 \to \mathbb{CP}^2$ be σ-process with center at z, $L = \sigma^{-1}(z)$. Denote again by $\pi : \mathbb{F}_1 \to \mathbb{CP}^1$ the composition $\pi \circ \sigma$.

Theorem 7.1. *Let two (generalized) cuspidal semi-algebraic curves B_1 and B_2 have the same braid monodromy factorization type $\Delta(B_1) = \Delta(B_2)$. Then there exists a smooth isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ such that*

1. *$F_{t|U}$ is the identity map for all $t \in [0, 1]$, where U is a neighbourhood of the exceptional curve L;*
2. *for each point $p \in B_1$ there exist neighbourhoods U_1 of p and $U_2 = F_1(U_1)$ of $F_1(p)$ with local complex coordinates (x_i, y_i) in U_i such that $\pi_{|U_i}$ is given by $(x_i, y_i) \mapsto x_i$ and $F_t^*(y_2) = \phi(x_1, y_1)$ is a smooth complex function holomorphic in y_1 ;*
3. *for each singular point $s \in B_1$ there exists a neighbourhood $U \subset \mathbb{F}_1$ of s such that $F_{1|U} : U \to F_1(U)$ is a holomorphic map;*
4. *$F_1(B_1) = (B_2)$.*

Corollary 7.2. *Let two (generalized) cuspidal semi-algebraic curves B_1 and B_2 have the same braid monodromy factorization type $\Delta(B_1) = \Delta(B_2)$. Then there exists a diffeomorphism of pairs $F : (\mathbb{CP}^2, B_1) \to (\mathbb{CP}^2, B_2)$ having properties* (2) *and* (3) *of Theorem* 7.1.

Proof. An isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ is called *compatible* with a semi-algebraic curve B (with respect to π) if $F_t(B)$ is a semi-algebraic curve B with respect to π for each $t \in [0,1]$.

The required isotopy F_t will be obtained as a composition of a sequence of smooth isotopies compatible with B_1.

Without loss of generality, we can assume that B_1 and B_2 are embedded into the same $\mathbb{F}_1$ and they are (generalized) cuspidal semi-algebraic curves of degree p with respect to π. We fix a point $\infty \in \mathbb{CP}^1$ such that $\pi^{-1}(\infty)$ is a generic fiber of π with respect to each B_1 and B_2. Denote by $\mathbb{C}^1 = \mathbb{CP}^1 \setminus \{\infty\}$ and $\mathbb{C}^2 = \mathbb{F}_1 \setminus (\pi^{-1}(\mathbb{C}^1) \cup L)$. We choose coordinates (x, y) in $\mathbb{C}^2$ such that $\pi : \mathbb{C}^2 \to \mathbb{C}$ is a projection on the first coordinate.

As in Section 6, let $K_i(x) = \{y \bigm| (x,y) \in B_i\}$, $i = 1,2$, ($K_i(x) =$ projection to y-axis of $\pi^{-1}(x) \cap B_i$), $N_i = \{x \bigm| \#K_i(x) \lneqq p\}$,

$$M_i' = \{(x,y) \in B_i \bigm| \pi \bigm|_{B_i} \text{ is not étale at } (x,y)\} \ (\pi(M_i') = N_i).$$

Let E_R (resp. D_R) be a closed disk of radius R with center at the origin o on x-axis (resp. y-axis) such that $M_i' \subset E_R \times D_R$, $N_i \subset \mathrm{Int}(E)$.

Assume that $\#K_i(o) = p$.

For each $u_{i,j} \in N_i$, $j = 1, \ldots, n$, we choose a disc $E_{i,j}$ of small radius $\epsilon \ll 1$ with center at $u_{i,j}$ and choose simple paths $T_{i,1}, \ldots, T_{i,n}$ connecting $u_{i,j}$ with o such that $< T_{i,1}, \ldots, T_{i,n} >$ is a bush. For each $j = 1, \ldots, n$ we choose a small tubular neighbourhood $U_{i.j}$ of $T_{i,j}$ and choose a disc E_o of radius $\epsilon \ll 1$ with center at o such that

$$U_{\Gamma_i} = (\bigcup_{j=1}^{n} U_{i,j}) \cup (\bigcup_{j=1}^{n} E_{i,j}) \cup E_o$$

is a tubular neighbourhood of a g-base $\Gamma_i = (l(T_{i,1}), \ldots, l(T_{i,n}))$ of the fundamental group $\pi_1(E_R \setminus N_i, o)$.

By Lemmas 6.1 and 6.2, we can assume that $E_{i,j} = \{\, x \in \mathbb{C}^1 \mid |x - x_{i,j}| < 2\}$, where $x_{i,j}$ is the coordinate of $u_{i,j}$.

Without loss of generality, we can assume that $T_{i,j} \cap E_{i,j}$ is a radius in $E_{i,j}$. We extend this radius to the diameter $d_{i,j}$ and denote by $\widetilde{T}_{i,j}$ the path continuing $T_{i,j}$ along this diameter, $T_{i,j} \subset \widetilde{T}_{i,j}$.

Step I. Since B_1 and B_2 have the same braid monodromy factorization type, then by Lemma 3.1 we can choose (and fix) g-bases Γ_i, $i = 1, 2$, such that the braid monodromy factorizations $\Delta(B_1)$ and $\Delta(B_2)$ associated to them are conjugation equivalent.

Lemma 7.3. *Let $B \subset \mathbb{F}_1$ be a semi-algebraic curve and let $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ be a smooth isotopy compatible with B such that F_t is the identity map over the complement of a neighbourhood $U \subset \mathbb{CP}^1$, $o \notin U$. Then B and $F_1(B)$ have the same braid monodromy factorization.*

Proof. A family of homomorphisms $\varphi_t : \pi_1(E \setminus N, o) \to B_p[\mathbb{C}_o, K]$ induced by F_t is continuous. Therefore, the braid monodromy factorizations corresponding to $F_t(B)$ do not depend on t, since $B_p[\mathbb{C}_o, K]$ is a discrete group.

By Lemmas 6.1, 6.2, and 7.3, there exists a smooth isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ having properties (1) – (3) of Theorem 7.1 and such that

(1) $F_1(U_{\Gamma_1} \times \pi^{-1}(U_{\Gamma_1})) = U_{\Gamma_2} \times \pi^{-1}(U_{\Gamma_2})$;
(2) $F_1(N_1 \times \pi^{-1}(N_1)) = N_2 \times \pi^{-1}(N_2)$;
(3) the types of singular points $s_{1,j} \in F_1(B_1)$ and $s_{2,j} \in B_2$ coincide over the point $x_{2,j} = x_j \in N_2 = N$;
(4) B_1 and $F_1(B_1)$ have the same (up to conjugation equivalence) braid monodromy factorization.

Denote again by B_1 its image $F_1(B_1)$.

Step II. By Lemma 6.9, there exist smooth isotopies $F'_t : \mathbb{F}_1 \to \mathbb{F}_1$ and $F''_t : \mathbb{F}_1 \to \mathbb{F}_1$ having properties (1) - (3) of Theorem 7.1, and such that $F'_1(B_1)$ and $F''_1(B_2)$ coincide over E_o, where E_o is a disc with center at the origin in $\mathbb{C}^1$.

Moreover, we can assume that $F_1'(B_1)$ and $F_1''(B_2)$ are constant sections over E_o:

$$K(u) = \{y_j \in D_R \mid y_j = 2j-1,\, j \in \mathbb{Z},\, 0 \le j \le p-1\}$$

for all $u \in E_o$.

Denote again by B_1 and B_2 their images $F_1'(B_1)$ and $F_1''(B_2)$, respectively.

Remark 7.4. *The isotopies F_t' and F_t'' change braid monodromy factorizations associated to B_1 and B_2 to conjugation equivalent factorizations, but they do not change their braid monodromy factorization types.*

We fix a frame $(\sigma_1, \ldots, \sigma_{p-1})$ of $B_p[\mathbb{C}_o^1, K(o)]$,

$$\sigma_j = [2j-3, 2j-1] = \{\, y \in \mathbb{C}^1 \mid 2j-3 \le \operatorname{Re} y \le 2j-1,\, \operatorname{Im} y = 0\,\}.$$

Step III. In $K_i(u_j) = \{q_{i,1}, \ldots, q_{i,p-1}\}$ let the point $q_{i,1}$ be the singular point of B_i, where u_j is the center of the disc $E_j = E_{1,j} = E_{2,j}$ defined in the definition of tubular neighbourhood of g-base.

Lemma 7.5. *Let $B = \{f(x,y) = 0\}$ be a germ of an analytic curve in $U = E_\varepsilon \times D_\varepsilon$ and let $(0,0)$ be a singular point of B of multiplicity 2 in direction $x = const$ (that is, $\#(B \cap (\{u\} \times D_1)) = 2$ for each $u \in E_\varepsilon$). Then there exists a smooth isotopy $F_t : U \to U$ such that*

1. *for all $t \in [0,1]$, $F_t = Id$ in a neighbourhood of the boundary of U;*
2. *there exists $\varepsilon_1 \ll \varepsilon$ such that $F_1(B) \cap V = \{y^2 - x^k = 0\}$, where $V = E_{\varepsilon_1} \times D_{\varepsilon_1}$;*
3. *$F_{t|V}$ is holomorphic for each t.*

Proof. By Weierstrass Preparation Theorem we can assume that B is given in U by

$$y^2 + h_1(x)y + h_2(x) = 0, \tag{1}$$

where $h_i(x)$ are analytic functions. Write (1) in the form

$$(y+\frac{1}{2}h_1(x))^2-(\frac{1}{4}h_1^2(x)-h_2(x))=(y+g_1(x))^2-x^kg_2(x))=0,$$

where $g_2(0)=c=re^{i\varphi}\neq 0$.

Let F'_t be an isotopy given by $F'_t(x,y)=(x,y+th(|x|)g_1(x))$, where $h(r)$ is a smooth monotone function such that $h(r)=1$ if $r<\varepsilon_1\ll\varepsilon$ and $h(r)=0$ if $r>\varepsilon-\varepsilon_1$,

One can show that a smooth map $\widetilde{F}''_t : E_{\varepsilon_1}\times[0,1]\to E_{\varepsilon}\times[0,1]$ given by

$$\widetilde{F}''_t(x)=x((1+(r-1)t)e^{it\varphi}+t(g_2(x)-c))^{1/k}$$

can be extended to a smooth isotopy $\widetilde{F}''_t : E_{\varepsilon}\times[0,1]\to E_{\varepsilon}\times[0,1]$ such that $\widetilde{F}''_t$ is the identity map in a neighbourhood of the boundary of E_{ε}. Then the composition $F_t=F''_t\circ F'_t$, where F''_t is given in U by

$$F''_t(x,y)=(\widetilde{F}''_t(x),y),$$

satisfies the conditions of Lemma 7.5.

By Lemmas 6.9 and 7.5, there exist smooth isotopies $F'_t:\mathbb{F}_1\to\mathbb{F}_1$ and $F''_t:\mathbb{F}_1\to\mathbb{F}_1$ having properties (1) - (3) of Theorem 7.1 and such that

(1) $F'_1(B_1)$ $F''_1(B_2)$ coincide over $\bigcup_{j=1}^{n}E'_j$, where $E'_j\subset E_{1,j}=E_{2,j}=E_j$ are some small neighbourhoods of $u_{i,j}=u_j$;

(2) by Lemmas 6.1 and 6.2, we can assume that $E'_j=\{\,x\in\mathbb{C}^1\mid |x-x_j|<2\,\}$, where x_j is the coordinate of the point u_j;

(3) in a neighbourhood of $(x_j,0)$ the curves $F'_1(B_1)$ and $F''_1(B_2)$ are given by the equation $y^2=(x-x_j)^k$;

(4) all other $p-2$ branches of $F'_1(B_1)$ (resp. $F''_1(B_2)$) are constant sections over E'_j and

$$K(u'_j)=\{\,y_1=-1,\dots,y_j=2j-3,\dots,y_p=2p-3\,\}.$$

for $u'_j=\{x'_j=x_j+1\}$.

Denote again by B_1 and B_2 their images $F_1'(B_1)$ and $F_1''(B_2)$ respectively. Without loss of generality, we can assume that $u_j' \in T_j$. Denote by u_j'' a point lying in the diameter $\widetilde{T}_j \cap E_j$ such that u_j'' is symmetric to u_j' with respect to the center u_j. Let d_j' be a part of the diameter connecting u_j' and u_j''.

Step IV. By Lemma 7.3 and Remark 7.4 the isotopies described above do not change the braid monodromy factorization types of B_1 and B_2. Write the braid monodromy factorizations of the curves B_1 and B_2 associated to the g-base Γ fixed above:

$$\Delta^2 = \prod_{j=1}^{n} Q_j^{-1}\sigma_1^{\nu_j}Q_j \quad \text{for } B_1;$$

$$\Delta^2 = \prod_{j=1}^{n} Q^{-1}Q_j^{-1}\sigma_1^{\nu_j}Q_jQ \quad \text{for } B_2.$$

We show that in our case there exists a smooth isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ having properties (1) - (3) of Theorem 7.1 and such that

(1) F_t is the identity map over the complement of the neighbourhood U_o of the point o;
(2) $F_1(B_1) = B_2$ over a neighbourhood $U_o' \subset U_o$;
(3) $F_1(B_1)$ and B_2 have the same braid monodromy factorization.

In fact, let

$$K = \{y_j \in \mathbb{C}^1 \mid y_j = 2j-3,\ j = 1, 2, \dots, p\}.$$

The half-twist $H_j = H(\sigma_j) \in B_p = B_p[\mathbb{C}^1_o, K(o)]$ can be represented by a geometric braid $\overline{\sigma}_j(s_1)$ in $\mathbb{C}^1 \times [0,1]$:

$$\overline{\sigma}_{j,l}(s_1) = l \ \text{ for } l = 1, \dots, j-1,\ j+2, \dots, p;$$

$$\overline{\sigma}_{j,j}(s_1) = e^{\pi(\beta(s_1)+1)i} + 2j - 2;$$

$$\overline{\sigma}_{j,j+1}(s_1) = e^{\pi\beta(s_1)i} + 2j - 2,$$

where $s_1 \in [0,1]$, $\beta(s_1)$ is real smooth monotone function such that $\beta(s_1) = 0$ for $s_1 \in [0, \frac{1}{3}]$ and $\beta(s_1) = 1$ for $s_1 \geq \frac{2}{3}$.

The element H_j^{-1} can be represented by a geometric braid $\overline{\sigma}_j^{-1}(s_1)$ in $\mathbb{C}^1 \times [0,1]$:

$$\begin{aligned} \overline{\sigma}_{j,l}^{-1}(s_1) &= l \ \text{ for } l = 1, \dots, j-1,\ j+2, \dots, p\,; \\ \overline{\sigma}_{j,j}^{-1}(s_1) &= e^{\pi(-\beta(s_1)+1)i} + 2j - 2\,; \\ \overline{\sigma}_{j,j+1}^{-1}(s_1) &= e^{-\pi\beta(s_1)i} + 2j - 2\,, \end{aligned}$$

The product $Q = H_{j_1}^{\delta_1} \cdot \ldots \cdot H_{j_k}^{\delta_k}$, where $\delta_l = \pm 1$, can be represented by the geometric braid $\overline{Q}(s_1)$ in $\mathbb{C}^1 \times [0,k]$:

$$\overline{Q}_l(s_1) = \overline{\sigma}_{j_m,l}^{\delta_{j_m}}(s_1 - m + 1) \ \text{ for } s_1 \in [m-1, m].$$

Let $U' \subset U_o$ be a neighbourhood of o for which there exists a diffeomorphism $\varphi : U' \to V = (-1, 2k+1) \times (0,2)$, $\varphi(o) = (0,0)$. Obviously, the paths T_j representing the bush can be chosen in such a way that $\varphi(T_j \cap U') \subset \{\, (v_1, v_2) \in V \mid v_1 < 0 \,\}$. Let $\alpha(r), r \geqslant 0$, be real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{4}{3}]$ and $\alpha(r) = 0$ for $r \geq \frac{5}{3}$. For $\overline{Q} = \sigma_{j_1}^{\overline{\delta}_1} \cdot \ldots \cdot \overline{\sigma}_{j_k}^{\delta_k}$ consider a smooth isotopy $F_{\overline{Q},t} : V \times E_R \to V \times E_R$, where

$$F_{\overline{Q},t} = F_{2k,\overline{\sigma}_{j_1}^{-\delta_1},t} \circ \cdots \circ F_{k+2,\overline{\sigma}_{j_{k-1}}^{-\delta_{k-1}},t} \circ F_{k+1,\overline{\sigma}_{j_k}^{-\delta_k},t} \circ F_{k,\overline{\sigma}_{j_k}^{\delta_k},t} \circ \cdots \circ F_{1,\sigma_{j_1}^{\delta_1},t}$$

and $F_{l,\sigma_j^{\delta_j},t}$ is given by functions

$$F_{l,\overline{\sigma}_j^{\delta_j},t}(s_1, s_2, y) = (s_1, s_2, f_{l,\overline{\sigma}_j^{\delta_j},t}(s_1, s_2, y),$$

where

$$f_{l,\overline{\sigma}_j^{\delta_j},t}(s_1, s_2, y) = \\ = \begin{cases} y\,, & s_1 \leq l-1\,; \\ 2j - 2 + (y - 2j + 2)e^{i\pi\delta_j\alpha(s_2)\beta(s_1 - l + 1)\alpha(|y - 2j + 2|)t}\,, & l - 1 \leq s_1 \leq l\,; \\ 2j - 2 + (y - 2j + 2)e^{i\pi\delta_j\alpha(s_2)\alpha(|y - 2j + 2|)t}\,, & s_1 \geq l \end{cases}$$

if $l \le k$, and

$$f_{l,\overline{\sigma}_j^{\delta_j},t}(s_1,s_2,y) = \\ = \begin{cases} y\,, & s_1 \ge l\,; \\ 2j-2+(y-2j+2)e^{i\pi\delta_j\alpha(s_2)\beta(l-s_1)\alpha(|y-2j+2|)t}, & l-1 \le s_1 \le l\,; \\ 2j-2+(y-2j+2)e^{i\pi\delta_j\alpha(s_2)\alpha(|y-2j+2|)t}, & s_1 \le l-1 \end{cases}$$

if $l \ge k+1$. One can check that

(1) $F_{\overline{Q},t}$ is the identity map over a neighbourhood of the boundary of V for all t;

(2) $F_{\overline{Q},1}(\overline{\mathcal{B}}) = \overline{Q}$, where $\overline{\mathcal{B}} = \{b_1(x(s_1,0)) \equiv 1,\dots,b_p(x(s_1,0) \equiv 2p-3\}$, $0 \le s_1 \le k$, is the trivial geometric braid.

(3) $F_{\overline{Q},1}(\mathcal{B}) = \{(s_1,s_2,-1),(s_1,s_2,1),\dots,(s_1,s_2,2p-3)\}$ are constant sections over

$$\{k-\frac{1}{3} < s_1 < k+\frac{1}{3}\} \times \{0 < s_2 < 2\};$$

Such isotopy $F_{\overline{Q},t}$ will be called *a $\overline{Q}$-twisting-untwisting of constant sections with support $\varphi^{-1}(V)$ and with center (V_0,z_0)*, where $V_0 = \varphi^{-1}(\{k-\frac{1}{3} < s_1 < k+\frac{1}{3}\} \times \{0 < s_2 < 2\}$ and $z_0 = \varphi^{-1}((k,0))$.

Let $\widetilde{F}_{\overline{Q},t} = \varphi^*(F_{\overline{Q},t})$. Denote again by B_1 its image $\widetilde{F}_{\overline{Q},1}(B_1)$.

In the notation of the definition of g-base Γ and its tubular neighbourhood, we change the g-base Γ to an equivalent one taking z_0 instead of o, changing each path T_j to a path starting at z_0 and coinciding with T_j outside the disc E_o. We change E_o to a disc $E_{z_0} \subset V_0$ with center at z_0 and choose new neighbourhoods U_j contained in the old neighbourhoods U_j. In the sequel, we denote again by o the point z_0.

By construction of the $\overline{Q}$-twisting-untwisting $F_{\overline{Q},t}$, the braid monodromy factorization of the curve B_1 will be

$$\Delta^2 = \prod_{j=1}^{n} Q^{-1}Q_j^{-1}\sigma_1^{\rho_j}Q_jQ,$$

that is, the braid monodromy factorizations of the curves B_1 and B_2 coincide.

Step V. Let u'_j be the point chosen in Step III, $u'_j \in T_j$, and let $U'_\Gamma \subset U_\Gamma$ be a tubular neighbourhood of the g-base Γ such that $U'_\Gamma \cap U(\partial U_\Gamma) = \emptyset$, where $U(\partial U_\Gamma)$ is a neighbourhood of the boundary ∂U_Γ of U_Γ.

Let us show that there exists a smooth isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ having properties (1) - (3) of Theorem 7.1 and such that

(1) F_t is the identity map over the complement of U_Γ ;

(2) $F_1(B_1) = B_2$ over $U'_\Gamma \subset U_\Gamma$.

To show it, for each j consider the geometric braids

$$\overline{\mathcal{B}}_1 = (b_{1,j,1}(x), \dots, b_{1,j,p}(x)) \text{ and } \overline{\mathcal{B}}_2 = (b_{2,j,1}(x), \dots, b_{2,j,p}(x)),$$

where x is moving along $T'_j \subset T_j$ starting at o and ending at u'_j. These geometric braids are representatives of elements $\beta_{i,j} \in B[\mathbb{C}_o, K]$, $i = 1, 2$. Since the corresponding factors of the braid monodromy factorizations for B_1 and B_2 coincide, then

$$\beta_{1,j}^{-1} H_1^{\nu_j} \beta_{1,j} = \beta_{2,j}^{-1} H_1^{\nu_j} \beta_{2,j} = Q_j^{-1} H_1^{\nu_j} Q_j.$$

Thus, $\beta_j = \beta_{1,j}\beta_{2,j}^{-1} \in C(H_1^{\nu_j})$. By Theorem 5.1 the element β_j can be written in the form

$$\beta_j = \mu_{j,1} \dots \mu_{j,k_j},$$

where each $\mu_{j,i}$ coincides with either $H_r^{\delta_{j,i}}$, where $\delta_{j,i} = \pm 1$ and $r = 1, 3, \dots, p$, or a full-twist $\Delta_{1,r}^{2\delta_{j,i}}$, $r = 3, \dots, p$, defined by the system of paths $(\sigma_1, \dots, \sigma_{p-1})$.

To each such β_j we associate a "twisting-untwisting" of $V_j \times \mathbb{C}^1$ similar to the one described in Step IV. Namely, we consider again $V_j = (-1, 2k_j + 1) \times (0, 2)$, and for each $\mu_{j,i}$ we define a smooth isotopy

$$F_{j,\mu_{j,i},t} : V_j \times \mathbb{C} \to V_j \times \mathbb{C},$$

and associate to $\beta_j = \mu_{j,1}\dots\mu_{j,k_j}$ the composition

$$F_{j,\beta_j,t} = F_{j,2k_j,\mu^{-1}_{j,1},t} \circ \dots \circ F_{j,k_j+2,\mu^{-1}_{j,k_j-1},t} \circ F_{j,k_j+1,\mu^{-1}_{j,k_j},t} \circ F_{j,k_j,\mu_{j,k_j},t} \circ \dots \circ F_{j,1,\mu_{j,1},t}$$

as follows. If $\mu_{j,l} = H(\sigma_r)^{\delta_{j,l}}$, then $F_{j,l,\mu_{j,l},t} = F_{l,\overline{\sigma}_r^{\delta_{j,l}},t}$ which was defined in Step IV (the number k in the definition of $F_{l,\overline{\sigma}_r^{\delta_{j,l}},t}$ is equal to k_j in our case). If $\mu_{j,l} = \Delta^{2\delta_{j,l}}_{1,r}$, then $F_{j,l,\mu_{j,l},t}$ is defined similarly, namely, it is given by

$$F_{j,l,\mu^{\delta_{j,l}},t}(s_1, s_2, y) = (s_1, s_2, f_{j,l,\mu^{\delta_{j,l}},t}(s_1, s_2, y),$$

where

$$\begin{aligned} &f_{j,l,\mu^{\delta_{j,l}},t}(s_1, s_2, y) = \\ &= \begin{cases} y\,, & s_1 \leq l-1\,; \\ r-2+(y-r+2)e^{2i\pi\delta_{j,l}\alpha(s_2)\beta(s_1-l+1)\gamma(|y-r+2|)t}\,, & l-1 \leq s_1 \leq l\,; \\ r-2+(y-r+2)e^{2i\pi\delta_{j,l}\alpha(s_2)\alpha(|y-r+2|)t}\,, & s_1 \geq l\,, \end{cases} \end{aligned}$$

if $l \leq k_j$, and

$$\begin{aligned} &f_{l,\mu^{\delta_{j,l}}_{j,l},t}(s_1, s_2, y) = \\ &= \begin{cases} y\,, & s_1 \geq l\,; \\ r-2+(y-r+2)e^{2i\pi\delta_{j,l}\alpha(s_2)\beta(l-s_1)\gamma(|y-r+2|)t}\,, & l-1 \leq s_1 \leq l\,; \\ r-2+(y-r+2)e^{2i\pi\delta_{j,l}\alpha(s_2)\gamma(|y-r+2|)t}\,, & s_1 \leq l-1\,, \end{cases} \end{aligned}$$

if $l \geq k_j+1$, where $\alpha(s)$, $\beta(s)$, and $\gamma(s)$, $s \geqslant 0$, are real smooth monotone functions such that $\alpha(s) = 1$ for $s \in [0, \frac{4}{3}]$ and $\alpha(s) = 0$ for $s \geq \frac{5}{3}$, $\beta(s) = 0$ for $s \in [0, \frac{1}{3}]$ and $\beta(s) = 1$ for $s \geq \frac{2}{3}$, and $\gamma(s) = 1$ for $s \in [0, r-1]$ and $\gamma(s) = 0$ for $s \geq r - \frac{1}{2}$.

Let us choose a neighbourhood $W_j \subset E_j$ containing the part of diameter d'_j connecting the points u'_j and u''_j, and such that there exists a diffeomorphism $\phi_j : W_j \to V_j$ such that $\phi_j(d'_j) = \{\, (s_1, s_2) \in V_j \mid 0 \leq s_1 \leq 2k_j,\ s_2 = 0\,\}$, $\phi_j(u_j) = (0,0)$. The diffeomorphism ϕ_j and the isotopy $F_{j,\beta_j,t}$ allow us to define a smooth isotopy $\widetilde{F}_{j,\beta_j,t} = (\phi_j^{-1} \times Id) \circ F_{j,\beta_j,t} \circ (\phi_j \times Id) : W_j \times \mathbb{C}^1 \to W_j \times \mathbb{C}^1$ which can be extended to a smooth isotopy such that $\widetilde{F}_{j,\beta_j,t}$ is the identity map outside $W_j \times \mathbb{C}^1$.

Let F_t be the composition of the constructed isotopies $\widetilde{F}_{j,\beta_j,t}$, $j = 1, \dots, n$. Denote again by B_1 its image $F_1(B_1)$ and by E_j a disc contained in $V_{0,j}$, where $V_{0,j}$ is the center of the "twisting-untwisting" $\widetilde{F}_{j,\beta_j,t}$, which is defined verbatim in Step IV. We choose a new point in $E_j \cap T_j$ and denote it again by u'_j. By construction of the isotopies $\widetilde{F}_{j,\beta_j,t}$, for each j the geometric braids

$$\overline{\mathcal{B}}_1 = (b_{1,j,1}(x), \dots, b_{1,j,p}(x)) \text{ and } \overline{\mathcal{B}}_2 = (b_{2,j,1}(x), \dots, b_{2,j,p}(x)),$$

where x is moved along $T'_j \subset T_j$ starting at o and ending at u'_j, are two representatives of the same element of $B_p[\mathbb{C}^1_o, K]$. Therefore by Lemma 6.10, there exists a smooth isotopy $F_t : \mathbb{F}_1 \to \mathbb{F}_1$ having properties (1) - (3) of Theorem 7.1 such that

(1) F_t is the identity map over the complement of the union of small neighbourhoods U_j of the paths T'_j;
(2) F_t is the identity map over $(\cup E_j) \cup E_o$;
(3) $F_1(B_1) = B_2$ over some tubular neighbourhood $U'_\Gamma \subset U_\Gamma$ of the g-base Γ.

Denote again by B_1 its image $F_1(B_1)$. The obtained curves B_1 and B_2 coincide over the tubular neighbourhood U'_Γ of the g-base Γ.

Step VI. The complement $\mathbb{P}^1 \setminus U'_\Gamma$ is simply connected. Let U_∞ be a simply connected neighbourhood of $\mathbb{P}^1 \setminus U'_\Gamma$ such that U_∞ is diffeomorphic to a disc and such that $u_j \notin U_\infty$ for all $j = 1, \dots, \#N$. Denote by $V = \pi^{-1}(U_\infty)$. Then $\pi : V \to U_\infty$ is a trivial fibering with fibres $\pi^{-1}(x) \simeq \mathbb{P}^1$ and $L \cap V$ is a section. Put $V_0 = V \setminus L$ and $\mathcal{B}_i = V_0 \cap B_i$, $i = 1, 2$. Then $V_0 \simeq U_\infty \times \mathbb{C}^1$ and $\pi_{|V_0}$ coincides with projection on the second factor. We apply Remark 6.7 and Lemma 6.9 to $\mathcal{B}_1$ and $\mathcal{B}_2$ to obtain a smooth isotopy F_t having all properties (1) - (4) of Theorem 7.1.

§8. Equivalence of braid monodromy factorizations and diffeomorphism types of surfaces.

In this Section we prove Theorem 2.

Let $F : (\mathbb{CP}^2, B_1) \to (\mathbb{CP}^2, B_2)$ be a diffeomorphism of pairs having the properties described in Corollary 7.2. This diffeomorphism induces an isomorphism $F^* : \pi_1(\mathbb{CP}^2 \setminus B_2) \to \pi_1(\mathbb{CP}^2 \setminus B_1)$.

By Proposition 1 in [Kul1], the set of non-equivalent generic morphisms of degree N with discriminant curve $B \subset \mathbb{CP}^2$ is in one-to-one correspondence with the set of epimorphisms from $\pi_1(\mathbb{CP}^2 \setminus B)$ to the symmetric group Σ_N satisfying some additional conditions (see details in [Kul1]). Since Chisini's Conjecture holds for $B_1 \subset \mathbb{CP}^2$, then there exists such a unique epimorphism from $\pi_1(\mathbb{CP}^2 \setminus B_1)$, which must coincide with the epimorphism $f_{1*} : \pi_1(\mathbb{CP}^2 \setminus B_1) \to \Sigma_N$ induced by f_1, where $N = \deg f_1$. Therefore, for B_2, there exists such a unique epimorphism, which must coincide with $f_{1*} \circ F^* = f_{2*} : \pi_1(\mathbb{CP}^2 \setminus B_2) \to \Sigma_N$. Consequently, the diffeomorphism $F : \mathbb{CP}^2 \setminus B_1 \to \mathbb{CP}^2 \setminus B_2$ can be lifted to a diffeomorphism $\Psi_0 : S_1 \setminus f_1^{-1}(B_1) \to S_2 \setminus f_2^{-1}(B_2)$.

In [Kul2], one can find a method how to reconstruct a surface S and a finite morphism $f : S \to \mathbb{CP}^2$ branched along $B \subset \mathbb{CP}^2$ if we know the homomorphism $f_* : \pi_1(\mathbb{CP}^2 \setminus B) \to \Sigma_N$. This method is based on the presentation S as N copies of $\mathbb{CP}^2$ with "standard cuts" pasted together along these cuts (to do such pasting together, we use the geometric desciption of the finite presentation $\pi_1(\mathbb{CP}^2 \setminus B)$ in terms of "shadows" and "screens" described in [Kul3]). Using this method, it is easy to see that the diffeomorphism Ψ_0 is uniquely extended to a homeomorphism $\Psi : S_1 \to S_2$.

Let $U \subset S_1$ be a neighbourhood of an ordinary cusp of B_1 such that $F_{|U}$ is holomorphic. It is well-known that if $f : X \to U$ is a three-sheeted covering of $U = \{ (x, y) \in \mathbb{C}^2 \mid |x| < 1, |y| < 1 \}$ branched along a curve given by $y^2 = x^3$ and such that f is not Galois covering, then such f is unique. Therefore, the homeomorphism Ψ is holomorphic (in particular, Ψ is smooth) in $f_1^{-1}(U)$. Similarly, Ψ is smooth in $f_1^{-1}(U)$, where U is a neighbourhood of a node of B_1 or a tangent

point of B_1 and a fiber of the projection π.

Let $z \in B_1$ be a non-singular point and let

$$U_1 \simeq \{\, (x_1, y_1) \in \mathbb{C}^2 \ \mid\ |x_1| < 1,\, |y_1| < 1 \,\}$$

be a neighbourhood of z in $\mathbb{CP}^2$, where (x_1, y_1) local holomorphic coordinates in $\mathbb{CP}^2$ such that $y_1 = 0$ is a local equation of B_1 and the projection π is given in U_1 by $(x_1, y_1) \mapsto x_1$. Similarly, let $U_2 = F(U_1)$ be a neighbourhood of $F(z)$ and let (x_2, y_2) be local holomorphic coordinates in U_2 such that $y_2 = 0$ is a local equation of B_2 and the projection π is given in U_2 by $(x_2, y_2) \mapsto x_2$. We have $x_2 = g_1(x_1)$ and $y_2 = g_2(x_1, y_1)$, where g_1 and g_2 are smooth functions and g_2 is holomorphic in y_1. Therefore g_2 can be writen in the form

$$g_2(x_1, y_1) = \sum_{n=1}^{\infty} a_n(x_1) y_1^n,$$

where all $a_n(x_1)$ are smooth and $a_1(x_1) \neq 0$ in U_1.

Each preimage $f_1^{-1}(U_1)$ and $f_2^{-1}(U_2)$ consists of $N-1$ connected components $U_{1,1}$,..., $U_{1,N-1}$ and $U_{2,1}, \dots, U_{2,N-1}$, respectively. Let f_1 (resp. f_2) is non-ramified in $\cup_{j=2}^{N-1} U_{1,j}$ (resp. in $\cup_{j=2}^{N-1} U_{2,j}$). Therefore, Ψ is smooth in $\cup_{j=2}^{N-1} U_{1,j}$. Besides, there exist local holomorphic cordinates (u_1, v_1) in $U_{1,1}$ (resp. (u_2, v_2) in $U_{2,1}$) such that f_1 is given in $U_{1,1}$ (resp. f_2 in $U_{2,1}$) by $y_1 = u_1^2$, $x_1 = v_1$ (resp. $y_2 = u_2^2$, $x_2 = v_2$). Consequently, Ψ is given by

$$\begin{aligned} u_2 &= u_1 \Big(\sum_{n=1}^{\infty} a_n(v_1) u_1^{2n-2}\Big)^{\frac{1}{2}} ; \\ v_2 &= v_1 \, . \end{aligned}$$

It is easy to see that

$$\Big(\sum_{n=1}^{\infty} a_n(v_1) u_1^{2n-2}\Big)^{\frac{1}{2}}$$

is a smooth function, since all $a_n(v_1)$ are smooth and $a_1(v_1) \neq 0$.

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